Complex Geometry

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Contents

1 Complex manifold and vector bundles

Reference:

Voisin, Introduction in Complex Algebraic Geometry, Cours Spécialisé.

Henri-Paul de Saint-Gervais, Uniformisation des Surfaces de Riemann.

1.1 Introduction and review

Locally modelled on $\mathbb{C}^n$. For the specific case when $n = 1$: Complex curves, Riemann surfaces. One endowed with structures locally modelled on $\mathbb{C}$. Holomorphic functions on $\mathbb{C}$ and on $\mathbb{C}^n$.

Definition 1.1. *A **holomorphic function** $f : U \subset \mathbb{C} \to \mathbb{C}$ is a differentiable map from $U \subset \mathbb{R}^n \to \mathbb{R}^n$ such that the differential commutes with multiplication by i (Complex linearity).*

Multiplication by i in coordinate x, y is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\frac{\partial}{\partial x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\frac{\partial}{\partial y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\begin{cases} i\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \\ i\frac{\partial}{\partial y} = -\frac{\partial}{\partial x} \end{cases}$. Then if df commutes with i, we have

$$\begin{cases} idf(\frac{\partial}{\partial x}) = df(i\frac{\partial}{\partial x}) = df(\frac{\partial}{\partial y}) \\ idf(\frac{\partial}{\partial y}) = df(i\frac{\partial}{\partial y}) = -df(\frac{\partial}{\partial x}) \end{cases} \iff \begin{cases} i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}, \\ i\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x} \end{cases} \quad (\textbf{Cauchy-Riemann Equations}).$$

$$\iff df = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = M_{a+bi}.$$

On $V \subset \mathbb{C}$, dx, dy are 1-forms.

$$dz = dx + idy, \quad d\overline{z} = dx - idy,$$

are 1-forms with values in $\mathbb{C}$. For $f : V \to \mathbb{C}$, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a 1-form with value in $\mathbb{C}$. $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$. Then we say f is holomorphic, iff df is $\mathbb{C}$-linear, iff $\frac{\partial f}{\partial \overline{z}} = 0$. Indeed, with notations

$$\begin{cases} \frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

$\frac{\partial f}{\partial \overline{z}} = 0$ iff $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$.

Theorem 1.1. *f is holomorphic iff f is analytic, meaning that for any $u \in U$, there is a power series $\sum\limits_{n \geq 0} a_n w^n$ of radius ρ such that for any $0 < r < \rho$ such that $D(u, r) \subset U$, and for any $z \in D(0, r)$, $f(u + z) = \sum\limits_{n \geq 0} a_n z^n$ with $a_n = \frac{f^{(n)}(u)}{n!}$.*

Proof. We only need to prove the only if part, and this is a consequence of **Cauchy formula**

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0}d\xi.$$

The holomorphic $\omega := \frac{f(\xi)}{\xi - z_0}d\xi$ on $U \setminus \{z_0\}$ is closed,

$$d\omega = \partial\omega + \overline{\partial}\omega = \frac{\partial}{\partial \xi}\left(\frac{f(\xi)}{\xi - z_0}\right)d\xi \wedge d\xi + \frac{\partial}{\partial \overline{\xi}}\left(\frac{f(\xi)}{\xi - z_0}\right)d\overline{\xi} \wedge d\xi = 0,$$

since $d\xi \wedge d\xi = 0$.

By Stocks formula

$$\int_\gamma \frac{f(\xi)}{\xi - z_0} d\xi = \int_{z_0 + \varepsilon \cdot e^{i2\pi t}} \frac{f(\xi)}{\xi - z_0} d\xi = \int_0^1 f(z_0 + \varepsilon \cdot e^{i2\pi t}) dt \xrightarrow{\varepsilon \to 0} f(z_0).$$

Now we use Cauchy formula to prove (roughly) that if f is holomorphic, we have f is analytic.

$$\begin{aligned} f(u+z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - u) - z} d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - u} \frac{1}{1 - \frac{z}{\xi - u}} d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - u} \sum_{n=0}^{\infty} (\frac{z}{\xi - u})^n d\xi \\ &= \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - u)^{n+1}} d\xi \Big) z^n. \end{aligned}$$

Note that the coefficient $\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - u)^{n+1}} d\xi = \frac{f^{(n)}(u)}{n!}$. □

For several variables $\mathbb{C}^n = \mathbb{R}^{2n}$, $(z_1, \cdots, z_n) \to (x_1, y_1, \cdots, x_n, y_n)$, the multiplication by i defines an operator $j \in \mathrm{End}(\mathbb{R}^{2n})$ with $j^2 = -\,\mathrm{id}$. Hence we have $i\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$.

Definition 1.2. *$f : U \subset \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function if f is differentiable and df commutes with multiplication by i.*

This is equivalent to, for any $1 \le k \le n$, $idf(\frac{\partial}{\partial x_k}) = df(i\frac{\partial}{\partial x_k}) = df(\frac{\partial}{\partial y_k})$, iff $i\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial y_k}$, iff $\frac{\partial f}{\partial \bar{z}_k} = 0$.

Theorem 1.2. *A function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic iff f is analytic, meaning that for any $u \in U$, there is a polydisk*

$$\{|z_1 - u_1| < R_1, \cdots |z_n - u_n| < R_n\} \subset U,$$

and a power series $\sum\limits_{i=(i_1,\cdots i_n), i_k \ge 0} a_i z^{i_1} \cdots z^{i_n}$ such that for any z with $|z_i| < R_i$, we have $f(u+z) = \sum\limits_i a_i z^{i_1} \cdots z^{i_n}$ and $\sum\limits_i |a_i| r_1^{i_1} \cdots r_n^{i_n} < +\infty$ fo any $r_k < R_k$.

Proof. This is a consequence of Cauchy formula

$$f(u) = \int_{|\xi_k - u_k| = r_k, \forall k} f(\xi) \frac{d\xi_1}{\xi_1 - u_1} \wedge \cdots \wedge \frac{d\xi_n}{\xi_n - u_n}.$$

□

Corollary 1.1. *If $U \subset \mathbb{C}^n$ is connected and $f : U \to \mathbb{C}$ is holomorphic and vanishes on a non-trivial open set $V \subset U$, then $f \equiv 0$ on U.*

Proof. Define

$$O := \{z \in U : \exists V(z) \text{ a neighborhood of } z \text{ such that } f|_{V(z)} \equiv 0\}.$$

O is not empty since $V \subset O$. Consider the set

$$F = \{z \in U : \frac{\partial^I f}{\partial z_I} = 0, \forall I = (i_1, \cdots, i_n), i_k \geq 0\}.$$

Since $F = O$ hence O is a non-empty closed and open set in U, hence $O = U$. □

Corollary 1.2 (**The maximal principle**). *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic map defined on the connected set U. Assume there is $u \in U$ and an open neighborhood $u \in V \subset U$ such that $|f(u)| \geq |f(z)|$ for any $z \in V$, then $f \equiv f(u)$ on U.*

Proof. Deduced from Cauchy formula

$$f(u) = \int_{|\xi_k - u_k| = \varepsilon_k, \forall k} f(\xi) \frac{d\xi_1}{\xi_1 - u_1} \wedge \cdots \wedge \frac{d\xi_n}{\xi_n - u_n},$$

with $\varepsilon_k > 0$ small enough such that $\{(z_1, \cdots, z_n) : |z_k - u_k| < \varepsilon_k\} \subset V$.

$$f(u) = \int_0^1 \cdots \int_0^1 f(u_1 + \varepsilon_1 \cdot e^{2i\pi t_1}, \cdots, u_n + \varepsilon_n \cdot e^{2i\pi t_n}) dt_1 \cdots dt_n.$$

Then

$$|f(u)| \leq \int_0^1 \cdots \int_0^1 |f(u_1 + \varepsilon_1 \cdot e^{2i\pi t_1}, \cdots, u_n + \varepsilon_n \cdot e^{2i\pi t_n})| dt_1 \cdots dt_n \leq |f(u)|.$$

Thus we have equalities in both inequalities.

First equality says $f(u_1 + \varepsilon_1 \cdot e^{2i\pi t_1}, \cdots, un + \varepsilon_n)$ have the same argument. The second equality says that $|f(u_1 + \varepsilon_1 \cdot e^{2i\pi t_1}, \cdots, un + \varepsilon_n| = |f(u)|$ for any $\varepsilon_1, \cdots, \varepsilon_n$. Hence f is constant $\equiv f(u)$ on the polydisk . Then by Corollary 1.1, $f \equiv f(u)$ on U. □

Definition 1.3. *$f : V \subset \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if $f = (f_1, \cdots, f_m)$ with f_k holomorphic, $\forall 1 \leq k \leq m$.*

If $m = n$, f is called a local biholomorphism if for any $u \in U$, there is $V(u) \subset U$ a neighborhood of u such that $f|_{V(u)} : V(u) \to f(V(u))$ is a holomorphic bijection with inverse which is holomorphic.

*A **biholomorphism** is a bijection which is a local biholomorphism at any point.*

Theorem 1.3 (**Constant Rank Theorem**). *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic map and assume $\exists u \in U$ such that on a neighborhood $V(u)$ of u, the rank of the differential $(\left(\frac{\partial f_i}{\partial z_l}\right)_{1 \leq l \leq n}^{1 \leq i \leq m})$ is constant equal to k.*

Then there exists a local biholomorphism between an open neighborhood of u in $V(u)$, called $W(u)$ and the polydisk $D^n = \{|z_i| < 1, \forall 1 \leq i \leq n\}$, $\varphi : W(u) \to D^n$ sends u to 0, and a local biholomorphism between $f(W(u))$ and D^m, ψ sends $f(u)$ to 0, such that $\psi \circ f \circ \varphi^{-1} : D^n \to D^m$ is $(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_k, 0, \cdots, 0)$.

Remark 1.1. *Particular case for $m = n = k$, local inverse theorem says that df has rank $k = m = n$ at a point u iff f is a local biholomorphism in the neighborhood of u.*

1.2 Typical examples with their automorphism group

Definition 1.4. *A complex manifold M is a topological Hausdorff space which admits a cover by open sets $(U_i)_{i\in\mathbb{N}}$ such that there exists $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ a collection of homeomorphism for any U_i and an open set $\varphi_i(U_i) \subset \mathbb{C}^n$, such that the transition map*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is biholomorphic.

The pair (U_i, φ_i) is called a local coordinate on M. The collection of $(U_i, \varphi_i)_{i\in I}$ is called an atlas. Two atlases are equivalent if their union is still an atlas. An equivalent class of atlases is the structure of a complex manifold. n is called the complex dimension of the manifold.

Example 1.1. *For $n = 1$ they are of complex dimension one (with this point of view they are complex curves). But they are real surface (real dimension 2) and they are called Riemannian surfaces.*

Example 1.2. *Open sets in $\mathbb{C}$ are complex manifolds. $D = \{z \in \mathbb{C} : |z| < 1\}$ is a complex manifold which is not biholomorphic to $\mathbb{C}$ (Liouville theorem). There are nice bijection between $\mathbb{D}$ and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. $F(z) = \frac{i-z}{i+z} : \mathbb{H} \to \mathbb{D}$ and $G(w) = i\frac{1-w}{1+w} : \mathbb{D} \to \mathbb{H}$.*

Example 1.3. *Riemann surface. $x^2 + y^2 + z^2 = 1$ is a Real manifold of dimension 2.*

$$\varphi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C} \cong \mathbb{R}^2, \quad \varphi_S : \mathbb{S}^2 \setminus \{S\} \to \mathbb{C} \cong \mathbb{R}^2; \quad \varphi_N \circ \varphi_S^{-1} : \mathbb{C}^* \to \mathbb{C}^*, z \mapsto \frac{1}{\bar{z}}.$$

is not holomorphic. But by setting $\varphi_S : \mathbb{S}^2 \setminus \{S\} \to \overline{\mathbb{C}} \cong \mathbb{R}^2$, we have the transition map $z \mapsto \frac{1}{z}$.

This is the same complex structure as $\mathbb{P}^1(\mathbb{C})$, the complex projective line:

$$\mathbb{P}^1(\mathbb{C}) = \{\textit{linear vector spaces of dimension } 1 \textit{ in } \mathbb{C}^2\}.$$

$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ if $\lambda \in \mathbb{C}^*$, hence $\mathbb{P}^1 \cong \mathbb{C}^2 \setminus \{0\}/\mathbb{C}^*$. An equivalence class is $[z_1 : z_2]$. On the open set $z_1 \neq 0$, $[z_1, z_2] \mapsto \frac{z_2}{z_1}$. On the open set $z_2 \neq 0$, $[z_1, z_2] \mapsto \frac{z_1}{z_2}$. Then the transition map is $z \mapsto \frac{1}{z}$ from $\mathbb{C}^*$ to $\mathbb{C}^*$.

Exercise 1.1. $\operatorname{Aut}(\mathbb{P}^1)$*: bijections which are holomorphic and with holomorphic inverse, are exactly those given by linear transformation of $\mathbb{C}^2$.*

Proof. Claim: any meromorphic function on $\mathbb{P}^1$ is rational, i.e. of the form $\frac{P}{Q}$ with $P, Q \in \mathbb{C}[X]$. In fact let $\alpha_1, \cdots, \alpha_n$ be poles of f in $\mathbb{C}$. Then there exists $k_1, \cdots, k_n \in \mathbb{N}$ such that $(z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} f$ is holomorphic on $\mathbb{C}$ with a possible pole at ∞, so it's a polynomial P, then f is rational.

If f is an automorphism, it has a unique pole and unique zero, so $f = \frac{az+b}{cz+d}$. $\square$

Example 1.4. *Take $\mathbb{C}^n$ and $(e_1, \cdots, e_{2n})$ a basis of the real vector space $\mathbb{R}^{2n}$. Then $\mathbb{C}^n/\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{2n}$ is a real manifold diffeomorphic to $(\mathbb{S}^1)^{2n}$ and it is also a complex manifold. A particular case of interest is $n = 1$, $\mathbb{C}/\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ is a complex structure on $\mathbb{S}^1 \times \mathbb{S}^1$. What are local coordinates here?*

Assume $x_0 \in \mathbb{C}$ and $r > 0$ such that $B(x_0, r)$ small enough such that all $\gamma \cdot B(x_0, r)$ are distinct when $\gamma \in \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ are transitions.

Exercise 1.2. *Let $X = \mathbb{C}/\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, with (e_1, e_2) a real basis of $\mathbb{R}^2$. Then X is biholomorphic to $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$. Moreover, $\tau, \tau' \in \mathbb{H}$ define the same complex structure iff $\exists g \in \mathrm{PSL}(2, \mathbb{Z})$ such that $g\tau = \tau'$ i.e. $\exists a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ such that $\frac{a\tau+b}{c\tau+d} = \tau'$.*

Proof. We admit the result $\mathrm{Aut}(\mathbb{C}) = \{\alpha z + \beta : \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\}$. If $f : \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \to \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau'$ is a biholomorphism, it induces a biholomorphism $\tilde{f} : \mathbb{C} \to \mathbb{C}$, hence there is $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$, such that $\tilde{f}(z) = \alpha z + \beta$. Moreover, we have $\tilde{f}(0) = 0$, hence $\beta = 0$.

We have $\alpha(\mathbb{Z} + \mathbb{Z}\tau) = \mathbb{Z} + \mathbb{Z}\tau'$, then there is $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{cases} c\alpha + d\alpha\tau = 1 \\ a\alpha + b\alpha\tau = \tau' \end{cases} \Rightarrow \alpha = \frac{1}{c + d\tau} \Rightarrow \tau' = \frac{a + b\tau}{c + d\tau}.$$

Since the map is invertible, by solving

$$A \cdot \begin{pmatrix} 1 \\ \frac{a+b\tau}{c+d\tau} \end{pmatrix} = \begin{pmatrix} \frac{1}{c+d\tau} \\ \frac{\tau}{c+d\tau} \end{pmatrix} \Rightarrow A \cdot \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \mathrm{id},$$

we say $\begin{pmatrix} c & d \\ a & b \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$, i.e. $ad - bc = \pm 1$. □

Definition 1.5. *Let X be a complex manifold and $f : X \to \mathbb{C}$ a continuous map. Then f is holomorphic iff for any local chart (U_i, φ_i) of X, $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic. This notion is independent of the chart because on $U_i \cap U_j$, we have $f \circ \varphi_j^{-1} = (f \circ \varphi_i^{-1}) \circ (\varphi_i \circ \varphi_j)$.*

Exercise 1.3. *If X is a connected compact complex manifold, any holomorphic map $f : X \to \mathbb{C}$ is constant.*

Proof. My proof: f is open.

Professor's proof. By maximal principle. Take $x_0 \in X$ such that $|f(x_0)| = \max_{z \in X} |f(z)|$. Then f admits a local maximal. Choose a locally chart (U_i, φ_i) around x_0 at x_0, then by maximal principle, $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is constant. Then $f \equiv f(x_0)$ in U_i.

Let O be the open set $\{z \in X : \exists V(z) s.t. f|_{V(z)} \equiv f(x_0)\}$. Then $U_i \subset \mathrm{O}$ hence O is not empty. But O is also the closed subset of $X = \{z \in X : f(z) = f(x_0), \frac{\partial f}{\partial z_I} = 0\}$. Then O is a non empty, closed and open set in X, hence $O = X$. □

Definition 1.6. *Let X and Y be complex manifolds. A continuous map $f : X \to Y$ is a holomorphic map, if for every chart (U_i, φ_i) of X and (W_j, ψ_j) of Y, we have*

$$\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i \cap f^{-1}(W_j)) \to \psi_j(W_j)$$

is holomorphic.

$f : X \to X$ is a biholomorphism if f is holomorphic, bijection and f^{-1} is holomorphic. It is enough to verify that f is holomorphic, bijective and $df(u)$ is invertible for any $u \in X$. The group of biholomorphisms of X is also called $\mathrm{Aut}(X)$.

Exercise 1.4. *(1)* $\mathrm{Aut}(X)$ *with* $X = \mathbb{P}^1(\mathbb{C})$ *is*

$$\mathrm{PSL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) : ad-bc \neq 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} [z_1 : z_2] = [az_1+bz_2 : cz_1+dz_2] \right\}.$$

(2)

$$\mathrm{Aut}(\mathbb{H}^1) = \mathrm{PSL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) : ad-bc \neq 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} [z_1 : z_2] = [az_1+bz_2 : cz_1+dz_2] \right\}.$$

(3)

$$\mathrm{Aut}(\mathbb{C}) = \{f \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) : f(\infty) = \infty\} = \{az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

Proof.

(1) See Exercise 1.1.

(3) Claim: f must be a polynomial. If not, $g(z) = f(\frac{1}{z})$ has an essential singularity.

Casorati-Weierstrass theorem. If $g : \mathbb{C}^* \to \mathbb{C}$ has an essential singularity at 0, then g maps any neighborhood of 0 to a dense set. If not, there is $\alpha \in \mathbb{C}$, $\varepsilon > 0$ such that $g(U) \cap D(\alpha, \varepsilon) = \varnothing$, hence $h(z) = \frac{1}{g(z)-\alpha}$ is bounded around 0 so holomorphic on U. Then $g(0) = \frac{1}{h(0)} + \alpha$, impossible.

Look at $V = \{z : |z| > 12\}$, then $f(V) \subset \mathbb{C}$ is dense. But $f(\{z : |z| < 12\}$ is open, f can't be injective! So f must be a polynomial. And f must be of degree 1 by injectivity.

(2) We prove that $\mathrm{Aut}(\mathbb{D}^2) = \mathrm{PSL}(2,\mathbb{R})$.

Fix $a \in D$, consider $\varphi_a(z) = \frac{z-a}{1-\bar{a}z} \in \mathrm{Aut}(D)$. Given $f \in \mathrm{Aut}(D)$, $a := f(0)$ consider $g = \varphi_a \circ f \in \mathrm{Aut}(D)$, $g(0) = 0$.

Schwarz's lemma. $f : D \to D$ holomorphic with $f(0) = 0$, then for any $z \in D$, $|f(z)| \leq |z|$. Moreover, if equality holds at a point, then there is $\theta \in \mathbb{S}^1$ such that $f(z) = e^{i\theta} z$.

$|g(z)| \leq |z|$ and $|g^{-1}(z)| \leq |z|$. Then equality holds.

□

1.3 Quotient space

Exercise 1.5. *Let X be a complex manifold and $\Gamma \subset \mathrm{Aut}(X)$ which* ***acts on X properly and discontinuously****, meaning that for any K_1, K_2 compact sets in X,*

$$\#\{\gamma \in \Gamma : \gamma \cdot K_1 \cap K_2 \neq \varnothing\} < +\infty,$$

and Γ acts without fixed points: if $\gamma \cdot x = x$ for some $x \in X$ then $\gamma = \mathrm{id}$.

Then X/Γ is a complex manifold and $X \to X/\Gamma$ is a local biholomorphism.

Exercise 1.6. *Let $\mathbb{C}^2 \setminus \{0\} = X$ and $\Gamma = \langle 2 \rangle : (z_1, z_2) \sim (2z_1, 2z_2)$. Then $\mathbb{C}^2 \setminus \{0\}/\langle 2 \rangle$ is a* ***Hopf manifold*** *diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$ and endowed with a complex structure (Here we treat $\mathbb{S}^1 \times \mathbb{S}^3$ as a real manifold, hence we can endow $\mathbb{S}^1 \times \mathbb{S}^3$ with this complex structure).*

Proof: from the polar coordinate, $\mathbb{R}^4 \setminus \{0\} =]0, \infty[\times \mathbb{S}^3$.

More results is in Example 1.6

Definition 1.7. *Let X and Y be complex manifolds and $f : X \to Y$ be a holomorphic map. Then f is called a* ***submersion*** *if $\forall x \in X$, $\mathrm{rank}(df(x)) = \dim Y$ (hence $\dim X \geq \dim Y$). f is called* ***immersion*** *if $df(x)$ is injective at any $x \in X$ (hence $\dim X \leq \dim Y$).*

Definition 1.8. *Let X be a complex manifold and $V \subset X$ be a subset. Then V is called a* ***complex submanifold*** *of X if for any $v \in V$ there exists an open set $v \in U \subset X$ and a holomorphic submersion $\varphi : U \to D^k$ such that $U \cap V = \varphi^{-1}(\{0\})$.*

In local coordinates, by the constant rank theorem we have (ψ_i) such that

$$(\varphi \circ \psi_i^{-1})(z_1, \cdots, z_n) = (z_1, \cdots, z_k).$$

In these coordinates, the subset $V \cap U$ is defined as $\psi_i^{-1}(D^n \cap \{z_1 = \cdots = z_k = 0\})$. It is a local chart proving that the complex submanifold V is a complex manifold of the dimension $n - k$. We call k the codimension of V in X.

Construction of submanifolds

Theorem 1.4. *Let $f : X \to Y$ be a holomorphic map between two complex manifolds. Let $y \in Y$ and assume that $\forall x \in f^{-1}(\{y\})$, the rank of $df(x)$ is the dimension of Y. Then $f^{-1}(\{y\})$ is a complex submanifold of X of dimension $\dim X - \dim Y$.*

Proof. This is an application of constant rank theorem. Let $x \in f^{-1}(\{y\})$ and in local coordinates (U_i, φ_i) in the neighborhood of x and (W_j, ψ_j) in the neighborhood of y, we have $(\psi_j \circ f \circ \varphi_i^{-1})(z_1, \cdots, z_n) = (z_1, \cdots, z_m)$. Locally $f^{-1}(u)$ is parametrised in the chart φ_i by $(0, \cdots, 0, z_{m+1}, \cdots, z_n)$. Then $f^{-1}(\{y\})$ is a submanifold of dimension $n - m$. □

Example 1.5. *$\{x^2 + y^2 + z^2 = 1\}$ is a submanifold codim 1 in $\mathbb{R}^3$.*

Exercise 1.7. *The only compact submanifold in $\mathbb{C}^n$ are points.*

Proof. Let V be a connected compact submanifold of $\mathbb{C}^n$. For any coordinate z_k, $z_k|_V : V \to \mathbb{C}$ is a holomorphic function on V. Since V is compact, z_k is constant hence V is a set of one point. □

Definition 1.9. *Here we ignore the definition of* ***complex projective space****, but we emphasis that the projection from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{P}^n(\mathbb{C})$ is holomorphic.*

Example 1.6. *Recall that we defined Hopf manifold in Example 1.6 as $\mathbb{C}^{n+1} \setminus \{0\}/(z_1, \cdots, z_{n+1}) \sim (2z_1, \cdots, 2z_{n+1})$. The map $M := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{Z} \to \mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$ is holomorphic and the fibers are identified with $\mathbb{C}^*/\mathbb{Z}$ (here $\mathbb{Z}$ is generated by $\times 2$).*

The exponential map $\mathbb{C} \to \mathbb{C}^*$ is a universal cover. $\exp(z_1) = \exp(z_2) \iff z_1 = z_2 + 2ik\pi$*, hence* $\exp : \mathbb{C}/2i\pi\mathbb{Z} \cong \mathbb{C}^*$ *is a biholomorphism. Then* $\exp : \mathbb{C}/2i\pi\mathbb{Z} + \ln 2\mathbb{Z} \cong \mathbb{C}^*/\langle 2 \rangle$*. The fibers of the projection $\mathbb{C} \to \mathbb{C}^*/\langle 2 \rangle$ are the elliptic curves* $\exp : \mathbb{C}/2i\pi\mathbb{Z} + \ln 2\mathbb{Z}$*.*

1.4 Complex projective manifolds

Definition 1.10. *Compact submanifolds in $\mathbb{P}^n(\mathbb{C})$ (they are called* ***complex projective manifolds****).*

Proposition 1.1. *$V \subset \mathbb{P}^n(\mathbb{C})$ is a complex projective manifold if V is a submanifold in $\mathbb{P}^n(\mathbb{C})$ and there are $f_1, \cdots, f_k$ homogeneous polynomials in $\mathbb{C}[x_0, \cdots, x_n]$ such that*

$$V = \big\{[z_0, \cdots, z_n] \in \mathbb{P}^n(\mathbb{C}) : f_l(z_0, \cdots, z_n) = 0, \forall l \in \{1, \cdots, k\}\big\}.$$

There is a theorem of Chow proving that any complex submanifold in $\mathbb{P}^n(\mathbb{C})$ is a complex projective manifold (GAGA Principal).

Example 1.7. *Let f be an irreducible homogeneous polynomial in $\mathbb{C}[x, y, z]$ such that*

$$\{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) = \frac{\partial f}{\partial y}(x, y, z) = \frac{\partial f}{\partial z}(x, y, z) = 0\} = \varnothing.$$

Then $V = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) : f(x, y, z) = 0\}$ is a 1-dimensional submanifold and hence a complex projective curve in $\mathbb{P}^2(\mathbb{C})$.

Proof. Euler formula, if f is homogeneous of degree m, $f(\lambda x, \lambda y, \lambda z) = \lambda^m f(x, y, z)$, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = mf.$$

Indeed, take derivative with respect to λ,

$$x\frac{\partial f}{\partial x}(\lambda x, \lambda y, \lambda z) + y\frac{\partial f}{\partial y}(\lambda x, \lambda y, \lambda z) + z\frac{\partial f}{\partial z}(\lambda x, \lambda y, \lambda z) = m\lambda^{m-1} f(x, y, z).$$

Then take $\lambda = 1$.

Take $p = [x, y, z]$ such that $z \neq 0$. We check the coordinates in the neighborhood of p such that $(u = \frac{x}{z}, v = \frac{y}{z})$.

Assume $f(p) = 0$, then $f(u, v, 1) = 0$ in coordinates (u, v). There is at lest one of the derivates $\frac{\partial f}{\partial u}(u, v, 1)$ and $\frac{\partial f}{\partial v}(u, v, 1)$ is non zero. Indeed if by contradiction,

$$f(u, v, 1) = \frac{\partial f}{\partial u}(u, v, 1) = \frac{\partial f}{\partial v}(u, v, 1) = 0,$$

by homogeneous, we get $f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) = \frac{\partial f}{\partial z}(x, y, z) = 0$ and by Euler formula $\frac{\partial f}{\partial z}(x, y, z) = 0$, a contradiction!

The same proof implies that for any irreducible homogeneous polynomial $f \in \mathbb{C}[x_0, \cdots, x_n]$ such that $\square$

More general: Assume $f_1, \cdots, f_k$ are homogeneous polynomials in $\mathbb{C}[x_0, \cdots, x_n]$ such that rank $\left(\frac{\partial f_i}{\partial z_j}\right)_{1 \leq i \leq k, 1 \leq j \leq n}$ is r at each point. Moreover if assume $r = k$, thus it is enough to assume that $r = k$ at points on the vanishing set. (Constant rank)

Then

$$\overline{V} = \left\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} : f_l(z_0, \cdots, z_n) = 0, \forall l \in \{1, \cdots, k\}\right\}$$

is a complex submanifold of codimension r in $\mathbb{C}^{n+1} \setminus \{0\}$ such that $(z_0, \cdots, z_n) \in \overline{V} \iff (\lambda z_0, \cdots, \lambda z_n) \in \overline{V}, \forall \lambda \in \mathbb{C}^*$.

Then $V = \pi(\overline{V})$, with $\pi :$ is such that

$$V = \left\{v \in \mathbb{P}^n(\mathbb{C}) : f_l(v) = 0, \forall l \in \{1, \cdots, k\}\right\}$$

is a complex projective manifold of codimension r in $\mathbb{P}^n(\mathbb{C})$.

1.5 Real and complex vector bundles

To any real or complex manifold X, one associates a canonical manifold TX which is its tangent space and has dimension $2 \dim X$. TX is an example of vector bundle over X.

Definition 1.11. *Let X be a manifold (could be real or complex). A real (complex) vector bundle over X is a manifold E endowed with a submersion $\pi : E \to X$ such that there is an open cover $(U_\alpha)_{\alpha \in I}$ of X by the open sets U_α with the property that for all $\alpha \in I$, there is $\tau_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$ a diffeomorphism such that $p_1 \circ \tau_\alpha = \pi$.*

Moreover if $U_\alpha \cap U_\beta \neq \varnothing$, then $\pi^{-1}(U_\alpha \cap U_\beta) \subset \pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\alpha \cap U_\beta) \subset \pi^{-1}(U_\beta)$,

$$\tau_\beta \circ \tau_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n, (u, v) \mapsto (u, g_{\beta\alpha}(u)v),$$

where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n; \mathbb{R})$.

Remark 1.2. $\tau_\alpha \circ \tau_\beta^{-1} = (\tau_\beta \circ \tau_\alpha^{-1})^{-1}$, *and hence $g_{\alpha\beta}(u) = g_{\beta\alpha}^{-1}(u)$.*

If $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, we have $g_{\alpha\gamma} \cdot g_{\gamma\beta} \cdot g_{\beta\alpha} = 1$. Also $g_{\alpha\alpha} = 1$. We will see those conditions define "1-cocycle" with values in $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ in sheaf theory.

Definition 1.12. *If X is a complex manifold and E is a complex vector bundle such that the transition cocycle $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$ is a holomorphic map. Then E is complex* *manifold and $\pi : E \to X$ is holomorphic. This is called* ***holomorphic complex vector bundle****.*

A map $s : X \to E$ is called a section if $\pi \circ s = \mathrm{id}$. The section is holomorphic if E is a holomorphic bundle, and $s : X \to E$ is holomorphic as maps between two complex manifolds. The space of sections of a vector bundle is a vector space.

Definition 1.13. *A rank n holomorphic vector bundle is trivial iff it admits n linearly independent (global) holomorphic sections.*

Remark 1.3. *Notice that not all holomorphic vector bundles admit holomorphic (global) sections. But all of them admit (local) sections over the sets $(U_\alpha)_{\alpha\in I}$.*

By definition $\pi^{-1}(U_\alpha) \xrightarrow{\tau_\alpha} U_\alpha \times \mathbb{C}^n$ holomorphic and $\tau_\alpha^{-1} \circ s_\alpha$ is a section of $\pi^{-1}(U_\alpha)$, where $s_\alpha(u) = \big(u, (1, 0, \cdots, 0)\big)$. Moreover, $\tau_\alpha^{-1} \circ s_\alpha|_{U_\alpha}$ will never vanish.

Definition 1.14. *Two vector bundles $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} X$ are isomorphic if there exists $f : E_1 \to E_2$ a diffeomorphism with $\pi_2 \circ f = \pi_1$ and for any $x \in X$, $f : \pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{x\})$ is a vector space isomorphism. i.e. at each $x \in X$, f gives an isomorphism between $E_{1,x}$ and $E_{2,x}$.*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Construction of holomorphic vector bundle

Example 1.8. *Assume X is a real manifold with an atlas $(U_i, \varphi_i)_{i\in I}$ then the real vector bundle over X defined by the cocycle $U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})$, $u \mapsto d(\varphi_j \circ \varphi_i^{-1})(\varphi_i(u))$ is the real tangent bundle TX of X. It is a manifold of dimension $2 \dim X$.*

*This bundle is isomorphic to the bundle of 1****-jets of maps*** *from $\mathbb{R}$ into X given by the following geometric construction.*

E as being the space of curves: $\gamma :]-\varepsilon, \varepsilon[\to X$ and $\gamma_1 \sim \gamma_2$ if $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$ is true in a local coordinates, this will be true in any other local coordinate. $E = \{\gamma\}/\sim$ (1-jets of curves), $E \to X$, $[\gamma] \mapsto \gamma(0)$.

Exercise 1.8. *Verify TX is given by the previous cocycle $d(\varphi_j \circ \varphi_i^{-1})$.*

Assume X is complex manifold with local charts (U_i, φ_i). Then the cocycle $d(\varphi_j \circ \varphi_i^{-1})$ is with values in $\mathrm{GL}(n, \mathbb{C})$ and holomorphic. It defines a holomorphic vector bundle of rank r over X, called the **holomorphic vector bundle** TX.

Another construction of TX is given by the 1-jets of maps from $\mathbb{C}$ to X. We will say that the holomorphic map

$$\gamma_1 : D(0, \varepsilon) = \{|z| < \varepsilon\} \to X$$

is equivalent to γ_2 iff $\begin{cases} \gamma_1(0) = \gamma_2(0) \\ \gamma_1'(0) = \gamma_2'(0) \end{cases}$. Then $\{\gamma\}/\sim$ is a complex vector bundle E over X through the map $E \to X$, $[\gamma] \mapsto \gamma(0)$ and it is isomorphic to TX.

Vector fields and 1-forms

If $f : X \to Y$ is a differentiable map, then $df : TX \to TY$ is a differentiable map.

A section of TX is called a vector field on X. Locally a vector filed is given by $f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ where f_i are smooth local functions.

If $f : X \to Y$ is a holomorphic map from X a complex manifold to a complex manifold Y, $df : TX \to TY$ between holomorphic tangent spaces.

A holomorphic section of TX is called a **holomorphic vector field**. Locally it is given by $\sum_{k=1}^{n} f_k(z_1, \cdots, z_n) \frac{\partial}{\partial z_k}$ with f_k holomorphic function.

The transition map of $\mathbb{P}^1(\mathbb{C})$ is $z \mapsto \frac{1}{z}$ hence the cocycle of $T\mathbb{P}^1(\mathbb{C})$ is given by $d(\frac{1}{z}) = -\frac{1}{z^2}$,

$$
\begin{aligned}
U_1 \times \mathbb{C} &\to U_2 \times \mathbb{C} \cong U_2 \times \mathbb{C} \\
(z, v) &\mapsto (\frac{1}{z}, -\frac{1}{z^2} v) \mapsto (\frac{1}{z}, \frac{1}{z^2} v)
\end{aligned}
$$

Exercise 1.9. *Find all global sections of $T\mathbb{P}^1(\mathbb{C})$.*

Proof. Assume we have a section $s_1 = f(z)\frac{\partial}{\partial z}$ on U_1 and $s_2 = g(w)\frac{\partial}{\partial w}$ on U_2. On $U_1 \cap U_2$, we have

$$f(z)\frac{\partial}{\partial z} = f(\frac{1}{w})\frac{\partial w}{\partial z}\frac{\partial}{\partial w} = -f(\frac{1}{w})w^2 \frac{\partial}{\partial w}.$$

Then on $U_1 \cap U_2$, we shall have

$$-w^2 f(\frac{1}{w}) = g(w).$$

Consider the power series of f and g. Since both f and g are holomorphic on $\mathbb{C}$, we say they don't have negative degree part. Hence the maximal degree for f is 2.

Moreover, given $s_1(z) = f(z)\frac{\partial}{\partial z} = (az^2 + bz + c)\frac{\partial}{\partial z}$, by setting $s_2 = -(a + bw + cw^2)\frac{\partial}{\partial w}$, we get a global section.

Then we finally proved that $\Gamma(T\mathbb{P}^1)$ is spanned by $\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}$, with dimension 3. □

Definition 1.15. *If E is a vector bundle over X, its* ***dual vector bundle*** *E^* is defined as being the vector bundle associated to the cocycle ${}^t(g_{\beta\alpha})^{-1}$ where $g_{\beta\alpha}$ is the cocycle defining E.*

$(E_x)^* = (\pi^{-1}(\{x\}))^*$ will be the fibers of E^* over $\{x\}$.

Remark 1.4. *Note that the action of $g \in \mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ changes to ${}^t g^{-1}$ when we associated the action on $(\mathbb{R}^n)^*$ ($^{-1}$ is due to the reverse direction and ^{t} is because we change the column vector into a row vector).*

In particular $(TX)^*$ is the vector bundle over X for which the sections are 1-forms. A local section of X is given by $\sum_{k=1}^{n} f_k dx_k$, where f_k is a smooth function. If X is a complex manifold, $(TX)^*$ will be a holomorphic vector bundle for which the local holomorphic sections will be given by $\sum_{k=1}^{n} f_k(z_1, \cdots, z_n) dz_k$ where f_k is a holomorphic function. $(TX)^*$ holomorphic cotangent bundle, also denoted by Ω^1_X.

Example 1.9. *$\mathbb{P}^1(\mathbb{C})$, $(T\mathbb{P}^1(\mathbb{C}))^*$ given by the cocycle $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^2$.*

Exercise 1.10. *Find all the global sections of $(T\mathbb{P}^1(\mathbb{C}))^*$.*

Proof. Similar to the proof of Exercise 1.9, we say there is no nontrivial global sections of $(T\mathbb{P}^1)^*$. □

Definition 1.16. *An isomorphism from E to F, where F are holomorphic bundles over X, is a holomorphic section of $(E^*) \otimes F$ where the holomorphic section is at each point an isomorphism ($\mathrm{Hom}(E, F) \cong E^* \otimes F$).*

Example 1.10. *Assume L_1 and L_2 are line bundles (rank 1). Then L_1^* is given by the cocycle $(g^1_{\beta\alpha})^{-1}$, where $g^1_{\beta\alpha}$ is the cocycle. $L_1^* \otimes L_2$ is given by the cocycle $(g^1_{\beta\alpha})^{-1} \cdot g^2_{\beta\alpha}$.*

Proposition 1.2. *$L_1 \cong L_2$ iff there is a non vanishing holomorphic section of $L_1^* \otimes L_2 \cong \mathrm{Hom}(L_1, L_2)$ iff $L_1^* \otimes L_2$ is holomorphically trivial ($\cong X \times \mathbb{C}$).*

Theorem 1.5. *Let E be a holomorphic line bundle over a compact Riemann surface X. Then the space of holomorphic sections of E is a vector space of finite dimension.*

Example 1.11. *If $E = X \times \mathbb{C}$, then holomorphic sections of E are holomorphic maps from X into $\mathbb{C}$. By the maximal principle, those maps are constant. So the space of sections have dimension 1.*

Example 1.12. ***Tautological line bundle** τ.*

$$L \subset \mathbb{P}^1 \times \mathbb{C}^2 \setminus \{0\} = \{(x, l) : x \in \mathbb{P}^1, l \in \mathbb{C}^2 \setminus \{(0, 0)\}, l = [x]\}.$$

This defines a holomorphic line bundle over $\mathbb{P}^1(\mathbb{C})$ with cocycle $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z$. Indeed

$$[z : 1] \mapsto a(z, 1) = az(1, w).$$

Recall that for two line bundles L_1 and L_2 given by cocycles g^1_{UV} and g^2_{UV}, the line bundle $L_1 \otimes L_2$ is given by $g^1_{UV} \cdot g^2_{UV}$. In particular, $L_1^{\otimes m}$ is given by $(g^1_{UV})^m$. The inverse of L_1 is $L_1^{-1} \cong L_1^*$.

Thus $T\mathbb{P}^1 = \tau^{-2}$, and we write $\tau = o(-1)$, $o(m) = \tau^{-m}$, hence $T\mathbb{P}^1(\mathbb{C}) = o(2)$.

Exercise 1.11. *Prove that the space of holomorphic sections of $o(m)$ has dimension $m+1$ and the space of sections identifies with polynomials in on variable of degree $\leq m$.*

Proof. Similar to the proof of Exercise 1.9. □

Remark 1.5. *The space of holomorphic sections is a vector space called $H^0(X, L)$. We will see that $H^1(X, L)$ will also be of finite dimensional.*

Proposition 1.3. *The space of smooth sections is of infinite dimension.*

Proof. For the trivial bundle $X \times \mathbb{C}$, they are C^∞ maps from X into $\mathbb{C}$. Moreover for any line bundle L, consider a local trivialization: there is $U \subset X$ such that $L|_U \cong U \times \mathbb{C}$. Consider the section $s : U \to U \times \mathbb{C}$, $u \mapsto (u, 1)$ which gives a section of $L|_U$. Take $\rho : U \to \mathbb{R}^+$ a bump function with $\mathrm{Supp}(\rho) \subset U$. Then $\rho \cdot s$ will extend by zero outside U. $\square$

Let us restate Remark 1.5

Theorem 1.6. *Let E be a holomorphic line bundle over a compact Riemann surface X. Then $H^0(X, L)$ is a vector space of finite dimension.*

Proof. We prove, by Riesz theorem, that for some norm on $H^0(X, L)$, the ball of radius 1 is compact.

We will endow L with a hermitian metric, meaning that on each fiber L_x, $x \in X$, we have an inner product: $L_x \cong \mathbb{C}$, we take $|z|$. In a local trivialization $L|_U \cong U \times \mathbb{C}$, we consider $h = |z|$.

Let $X = \bigcup\limits_{\alpha \in I} U_\alpha$ such that $L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$, consider smooth functions $\rho_\alpha : U_\alpha \to \mathbb{R}^+$ such that $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$ is compact and $\sum\limits_{\alpha \in I} \rho_\alpha = 1$ (and locally finite).

Define the hermitian metric h on L as being $\sum\limits_{\alpha \in I} \rho_\alpha |z_\alpha|$, where $L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$. For any section $s \in H^0(X, L)$ we define $\|s\|_h = \max\limits_{p \in X} h(p)\big(s(p)\big)$. We want to prove that the unitary ball of $H^0(X, L)$ is compact.

Let $(s_n)_{n \geq 1} \subset H^0(X, L)$ such that $\|s_n\|_h \leq 1$. As before let $X = \bigcup\limits_{\alpha \in I} U_\alpha$ such that $L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$. For technical reason, consider $W_\alpha \subset V_\alpha \subset U_\alpha$ such that $\overline{W_\alpha} \subset V_\alpha$, $\overline{V_\alpha} \subset U_\alpha$ and $\bigcup\limits_\alpha W_\alpha = X$.

On U_α, there is a nowhere vanishing sections $s_\alpha \in H^0(U_\alpha, L)$ which trivializes $L|_{U_\alpha}$, $s_n|_{U_\alpha} = f_n \cdot s_\alpha$ for $f_n \in \mathcal{O}(U_\alpha)$.

$$1 \geq \|s_n|_{U_\alpha}\|_h \geq |f_n|_\infty \cdot \min_{p \in \overline{V_\alpha}} h(p)\big(s_\alpha(p)\big),$$

hence on V_α, $|f_n| \leq \frac{1}{\min\limits_{p \in \overline{V_\alpha}} h(p)\big(s_\alpha(p)\big)}$, i.e. f_n is bounded on V_α. Then $(f_n)_{n \geq 1}$ is an equicontinuous family on W_α. By Montel theorem, $(f_n)_{n \geq 1}$ admits a subsequence which converge on W_α. Then there is a unique $f_\infty^\alpha \in \mathcal{O}(W_\alpha)$ and a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$, such that $\lim\limits_{n \to \infty} f_{\sigma(m)}^\alpha = f_\infty^\alpha$ and the convergence is uniform on any compact set in W_α.

We can use the diagonal process to get that for any $\alpha \in I$, $\lim\limits_{n \to \infty} f_{\sigma(n)}^\alpha = f_\infty$. On $W_\alpha \cap W_\beta \neq \varnothing$, there is a cocycle condition $f_n^\alpha = g_{\alpha\beta} f_n^\beta$ hence $f_{\sigma(n)}^\alpha = g_{\alpha\beta} f_{\sigma(n)}^\beta$ hence $f_\infty^\alpha = g_{\alpha\beta} f_\infty^\beta$. The f_∞^α glue into a global section of $H^0(X, L)$, hence s_∞ is a limit of $s_{\sigma(n)}$. $\square$

2 Riemannian Surface

2.1 Definitions and Isothermal Coordinate

Definition 2.1. *Let V be a real vector space of dim 2, oriented. A **complex structure on** V is $j \in \mathrm{End}(V)$ such that $j \circ j = -\mathrm{id}$ and we will ask that j is compatible with the orientation, $\forall v \in V$, (v, jv) is a direct basis, we should think that $jv = i \cdot v$.*

Proposition 2.1. *The complex structure on V is equivalent with an inner product on V up to a resealing.*

Proof. Assume we have g, then define jv as being $jv \perp v$ and $\|jv\| = \|v\|$, then (v, jv) is a direct basis compatible with the orientation.

If g_1 and g_2 are two inner products, g_1 and g_2 define the same complex structure if $\exists \lambda \in \mathbb{R}^+$ such that $g_1 = \lambda g_2$.

Assume j is defined, then define g as the following inner product

$$\|v\| = \lambda, \|jv\| = \lambda, \langle v, jv \rangle = 0.$$

Thus define g up to a constant. $\square$

Definition 2.2. *Let S be a surface. We assume S is **oriented**, meaning there exists an atlas defining S such that the transition maps $\varphi_i \circ \varphi_j^{-1}$ have a differential which is positive* $\det(d(\varphi_i \circ \varphi_j^{-1}) > 0$.

Remark 2.1. *Any Riemann surface is oriented.*

Definition 2.3. ***A complex structure on Riemann surface** S compatible with the orientation is a smooth section $j \in \mathrm{End}(TS)$ such that $j \circ j = -\mathrm{id}$ and $\forall v \in TS$, (v, jv) is a direct basis.*

For a Riemann surface, multiplication by i: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ defines a complex structure.

Definition 2.4. *A **Riemannian metric** on S is a smooth section of $\mathrm{Sym}^2(T^*S)$: it is an inner product g_x on each tangent space T_xS, $\forall x \in S$, which is smooth with respect to $x \in S$ in the following way:*

If $\varphi : \mathbb{R}^2 \to S$, $0 \mapsto m$ is a local chart in a neighborhood of $m \in S$, $\varphi^* g = \sum_{i,j}^{n} g_{ij} dx_i dx_j$, meaning that, if $X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ are local vector fields, then $g(X, Y) = \sum_{i,j=1}^{n} a_i g_{ij} b_j$.

*Two Riemannian metrics g_1 and g_2 are **conformal** if $\exists \lambda : S \to \mathbb{R}^+$ such that $g_1 = \lambda g_2$.*

We will say that a Riemannian metric g on S is compatible with the complex structure if there exists local coordinates in which $g = \lambda(x, y)(dx^2 + dy^2)$. This condition means that the complex structure defined by g is the one given by $\times i$.

Proposition 2.2. *Let S be a Riemann surface. Then there exists a Riemannian metric on S which is compatible with the complex structure.*

Proof. Take holomorphic coordinates $\rho_i : U_i \subset \mathbb{C} \to \rho_i(U_i) \subset S$ and in each holomorphic coordinate consider $g_i = |z_i|^2 = x_i^2 + y_i^2$ and consider a partition of unity ρ_i: $g = \sum_{i=1}^{n} \rho_i g_i$. $\square$

On surfaces we have an important result:

Theorem 2.1 (**Isothermal coordinates (local)**). *Let g be a Riemannian metric on a surface S. Then for any $p \in S$, there exists a local chart $\varphi : U \subset \mathbb{R}^2 \to \varphi(U) \subset S$, $0 \mapsto p$, such that $\varphi^* g = \lambda(x, y)(dx^2 + dy^2)$, $\lambda : U \to \mathbb{R}^+$.*

Lemma 2.1. *Let g be a Lorentz metric on a surface S. Then for any $p \in S$, there exists local coordinates at p, such that $g = \lambda(x, y)(dx^2 - dy^2)$.*

Proof of the lemma. g_1 and g_2 two Lorentz metrics are conformal: $\exists \lambda : S \to \mathbb{R}^+$ $g_1 = \lambda g_2$ iff $\{v \in TS : g_1(v) = 0\} = \{v \in TS : g_2(v) = 0\}$. The standard $g_0 = dx^2 - dy^2$ admits two line fields of isotropic vectors: $\Delta_1 = \{x = y\}$ and $\Delta_2 = \{y = -x\}$.

Consider L_1 and L_2 the two isotropic lines of g and we want to identify them on Δ_1 and Δ_2.

Exercise 2.1. *X, Y, find local functions f, g such that $[fX, gY] = 0$, then there is a local change of coordinates, $\rho_*(fX) = \frac{\partial}{\partial x}$, $\rho_*(gY) = \frac{\partial}{\partial y}$.*

Proof of the exercise. Note that $\frac{X(g)}{g}Y - \frac{Y(f)}{f}X = [Y, X]$, hence we can solve out f and g. The existence of x, y is constructed by flow, or by baby version of Frobenius theorem. $\square$

$\square$

Proof of Isothermal coordinates by Gauss. Here we use the convention that

$$dxdy = \frac{1}{2}(dx \otimes dx + dy \otimes dy).$$

Complexify the metric and look to the isotopic lines in the complex domain.

$$g(x, y) = a(x, y)dx^2 + 2b(x, y)dxdy + c(x, y)dy^2$$

and think of it on an open set in $\mathbb{C}^2$. The same proof shows that you can rectify (find local coordinates) the metric on $\lambda(x, y)(dx - idy)(dx + idy) = \lambda(x, y)(dx^2 + dy^2)$.

$$g = \frac{1}{a}(adx + (b + i\sqrt{ac - b^2})dy)(adx + (b - i\sqrt{ac - b^2})dy) = \omega_1\omega_2.$$

Then ω_1 is a holomorphic 1-form on an open set in $\mathbb{C}^2$, $\operatorname{Ker}\omega_1$ defines a family of curves which are locally given by an equation $f(x, y) = constant$, $\operatorname{Ker}\omega_1 = \operatorname{Ker} df$. There is a function $h : U \to \mathbb{C}^*$, then $\omega_1 = hdf = h(du + idv)$.

On $\mathbb{R}^2$, $\omega_2 = \overline{\omega_1} = \overline{h}(du - idv)$.

$$g = \frac{1}{a}\omega_1\omega_2 = \frac{1}{a}h\overline{h}(du + idv)(du - idv) = \frac{1}{a}|h|^2(du^2 + dv^2).$$

$\square$

Corollary 2.1. *Let* $j \in \mathrm{End}(TS)$ *such that* $j \circ j = -\mathrm{id}$. *Then there exists local coordinates in which* j *is* $j_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Corollary 2.2. *Any oriented surface* S *admits complex structures.*

Proof. Consider a Riemannian metric g on S. Consider all oriented local coordinates where g is $\lambda(x,y)(dx^2 + dy^2)$. The transition maps are local diffeomorphisms, preserving the angles of the euclidien metric and preserving orientation, then they are holomorphic maps. □

Remark 2.2. *Similar to the vector space case, on Riemann surface,* g_1 *and* g_2 *will produce the same complex structure if* $\exists \lambda : S \to \mathbb{R}^+$ *such that* $g_1 = \lambda g_2$.

Proposition 2.3. $(T\mathbb{P}^1)^*$ *admits no sections.*

Proof. First proof. Let $\omega \in H^0(\mathbb{P}^1, T^*\mathbb{P}^1)$. $\omega = f(y)dy$, where f holomorphic. Then $-f(\frac{1}{y})\frac{1}{y^2}dy$ is holomorphic, hence $f \equiv 0$ (This proof is similar to Exercise 1.9).

Second proof. $\omega \in H^0(\mathbb{P}^1, T^*\mathbb{P}^1)$ and $X \in H^0(\mathbb{P}^1, T\mathbb{P}^1)$. Then $\omega(X) \in H^0(\mathbb{P}^1, \mathbb{C})$, since $\mathbb{P}^1$ is compact, $\omega(X)$ is constant. Since X has zeros on $\mathbb{P}^1$ (Harry ball), the constant is 0. On the open set where X is nowhere 0, we say ω is 0 everywhere, hence ω is 0 on X. □

Remark 2.3. *The second proof shows that a nontrivial line bundle cannot have holomorphic forms and holomorphic vector fields at the same time.*

2.2 Uniformization of Riemann surface

In this section we assume all the surface to be connected as a priori.

Theorem 2.2 (Riemann)**.** *Let* $U \subsetneq \mathbb{C}$ *be a simply connected, connected open set. Then there exists an biholomorphism* $\varphi : U \to \{z \in \mathbb{C} : |z| < 1\} = \mathbb{D}$.

Proof. In the devoir. □

Example 2.1. $U = \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, *then* $\varphi(z) = \frac{z-i}{z+i}$ *is a biholomorphism between* $\mathbb{H}$ *and* $\mathbb{D}$.

Remark 2.4. *By considering* $z^{\pi/\alpha}$, *every* ***sector*** *is biholomorphic to a half plane.*

Given a ***strip****, after a suitable rotation* ($z \mapsto \lambda z$)*, it is horizontal. Consider* $\eta \mapsto e^{\eta}$ *we get a sector.*

Remark 2.5. *There is not (by Poincaré)* $\varphi : \mathbb{C} \to \mathbb{D}$ *biholomorphism even if* $\mathbb{D}$ *and* $\mathbb{C}$ *have the "same" real structure.*

Proof. Any holomorphic $\varphi : \mathbb{C} \to \mathbb{D}$ is a constant. □

Theorem 2.3 (**Uniformization of Riemann surface**, Poincaré)**.** *Let S be a Riemann surface and assume that S is simply connected and connected. Then S is biholomorphic to $\mathbb{P}^1(\mathbb{C})$ (when the Riemann surface is compact), or to $\mathbb{C}$, or to $\mathbb{D}$.*

Remark 2.6. *$\mathbb{P}^1(\mathbb{C})$ and $\mathbb{D}$ don't have the same real structure, there are not homeomorphism since D is not compact. The same reason for $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{C}$.*

Proposition 2.4. *Recall*

$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}(2, \mathbb{C})$;

$\mathrm{Aut}(\mathbb{C}) = \{az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}$ *motions;*

$\mathrm{Aut}(\mathbb{D}) = \mathrm{PSL}(2, \mathbb{R})$.

Remark 2.7. *The biholomorphism gives a conformal map between S and the three basic model, whose curvature is $+1, 0, -1$ respectively.*

Particular case. Let g be a Riemannian metric on $\mathbb{R}^2/\mathbb{Z}^2$, then g is conformally equivalent to $dx^2 + dy^2$ (Find the universal covering).

For the hyperbolic case, the biholomorphic is indeed an isometry (this can be proved by direct calculation).

Corollary 2.3. *Let S be a compact and simply connected, therefore S is diffeomorphic to $\mathbb{S}^2$.*

Proof. A simply connected surface is orientable. Furthermore, we can endow it with a Riemannian metric g. Therefore S is endowed with a complex structure by the theorem of isothermal coordinate 2.1.

Then there is, by the uniformization theorem 2.3, a biholomorphism $\varphi : S \to U$, where $U = \mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or D. Since $\mathbb{P}^1(\mathbb{C})$ is the only compact model, we have $\varphi : S \to \mathbb{P}^1(\mathbb{C})$ hence S is biholomorphic to $\mathbb{S}^2$.

Furthermore, this shows that (S, g) is "uniformly" conformal equivalent to $\mathbb{S}^2$ with its canonical metric. □

Exercise 2.2. *Prove that any isometry that preserves g is in the orthogonal group* $\mathrm{O}(3, \mathbb{R})$.

Proof. Let f to be the isometry on $\mathbb{S}^2 \subset \mathbb{R}^3$. Define $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3, x \mapsto |x| \cdot f(\frac{x}{|x|})$, then extend it continuously to 0. Since $\mathbb{S}^2 \subset \mathrm{Im}\, f$, F is onto.

f is an isometry on $\mathbb{S}^2$, hence it preserves the distance in $\mathbb{S}^2$. Note that the distance is just the angle between two vectors with endpoints in $\mathbb{S}^2$, hence f preserves the angles. By the construction of F, we say $\triangle(0, x, y)$ and $\triangle(0, F(x), F(y))$ are two congruent triangle. Then

$$\langle F(x), F(y)\rangle = \langle x, y\rangle.$$

So F also preserves the inner product in $\mathbb{R}^3$.

$$\langle F(ax + by) - aF(x) - bF(y), F(z)\rangle = \langle ax + by, z\rangle - \langle ax, z\rangle - \langle by, z\rangle = 0, \forall x, y, z \in \mathbb{R}^3.$$

Then we have F is a linear map. Moreover it preserves the inner product, hence $F \in \mathrm{O}(3, \mathbb{R})$. □

Remark 2.8. *The topological result was know before the "uniformization theorem": Any simply connected surface is homeomorphic to* $\mathbb{S}^2$ *or* $\mathbb{R}^2$*.*

2.3 Complex structure and hyperbolic geometry model

We have three model of hyperbolic geometry.

$$\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta}\frac{z-a}{1-\overline{a}z} : \theta \in \mathbb{R}, a \in D\}.$$

By direct calculation, we say the biholomorphism on $\mathbb{D}$ preserves the metric $\frac{|dz|^2}{(1-|z|^2)^2}$ with negative constant curvature.

$\mathbb{D} \cong \mathbb{H}$ semi-plane model. $\mathrm{Aut}(\mathbb{H}) = \mathrm{Stab}_{\mathbb{H}}(\mathrm{Aut}(\mathbb{P}^1)) \cong \mathrm{PSL}(2,\mathbb{R})$. $\mathbb{H}$ with the Poincaré metric $\frac{dx^2+dy^2}{y^2}$ is isometric to $(D, \frac{dx^2+dy^2}{(1-x^2-y^2)^2})$ and $\mathrm{Aut}(\mathbb{H})$ preserves this metric.

Consider $\mathbb{R}^2$ with the standard orientation, **the set of complex structure compatible with the given orientation** is

$$\mathrm{Comp}^+(\mathbb{R}^2) = \{J \in \mathrm{End}(\mathbb{R}^2), J \circ J = -\mathrm{id}, (v, Jv) \text{ is an oriented basis}, \forall v \in \mathbb{R}^2\}.$$

In this case $i \times v = Jv$. We denote the standard example $\mathbb{R}^2 \cong \mathbb{C}$ by $J_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

What can we say about the eigenvalues of J? $J^2 + \mathrm{id} = 0$ gives that the eigenvalues of J are $\pm i$. J is similar to J_0 in $M_{2\times 2}(\mathbb{C})$, hence there exists $P \in \mathrm{SL}(2,\mathbb{R})$ such that $P \circ J \circ P^{-1} = J_0$.

1. **Hyperboloid Model:** $\mathrm{Comp}^+(\mathbb{R}^2)$ is a homogeneous space for the action of $\mathrm{SL}(2,\mathbb{R})$ by conjugacy. $\mathrm{Tr}\, J = 0$ and $\det J = 1$, hence

   $$\mathrm{Comp}^+(\mathbb{R}^2) = \{J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R}, \det J = -a^2 - bc = 1\}.$$

   $\det J = -a^2 - bc = -a^2 - (\frac{b+c}{2})^2 + (\frac{b-c}{2})^2$. Then $\det J = 1$ defines the "hyperboloid" in the space $\{x = a, y = \frac{b+c}{2}, z = \frac{b-c}{2}\}$ Given by the equation $-x^2 - y^2 + z^2 = 1$.

   The quadratic form $Q = -\det J = x^2+y^2-z^2$, then the hyperboloid will be $Q = -1$.

   $\det(PJP^{-1}) = \det J$, $\forall P \in \mathrm{SL}(2,\mathbb{R})$ then $Q = -\det$ is invariant by the action of $\mathrm{SL}(2,\mathbb{R})$ ($\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{O}(2,1)$).

   Q has $(2,1)$ as signature. However $\forall v \in$ hyperboloid, i.e. $Q(v) = -1$, we have

   $$T_v(Q^{-1}(\{-1\}) = v^{\perp Q}.$$

   Now we have that the space of complex structure on $\mathbb{R}^2$ is equivalent to a hyperboloid with the Lorentz metric.

2. **Upper half plane model:**

   $$\mathrm{Comp}^+(\mathbb{R}^2) = \{q : \mathbb{R}^2 \to \mathbb{R} : q \text{ symmetric positive-definite linear functor}\}/q \sim \lambda q.$$

Set $q = ax^2 + bxy + cy^2 \sim x^2 + bxy + cy^2$ for $b, c \in \mathbb{R}$, and we can see that q uniquely determines and also is uniquely determined by $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ (the equation must have imaginary roots), hence we conclude that the space of complex structure on $\mathbb{R}^2$ is equivalent to the upper half plane (maybe by calculation we say with the metric $\frac{dx^2+dy^2}{y^2}$).

3. **Disc model:** We can also write $g = \lambda|z + \Phi\overline{z}|^2$, $\lambda > 0$, $|\Phi| < 1$ (guarantees that the main part of g is $|z|$). Here $|\Phi| < 1$ is because if $|\Phi| > 1$, we have

$$g = \lambda|z + \Phi\overline{z}|^2 = \lambda|\overline{z} + \overline{\Phi}z|^2 = \lambda|\Phi| \cdot |z + \frac{1}{\overline{\Phi}}\overline{z}|^2.$$

Hence g is equivalent to $|z + \frac{1}{\Phi}\overline{z}|^2$ with $|\frac{1}{\Phi}| < 1$. And by direct calculation, $|\Phi|$ cannot be 1. Thus Φ gives the model of the disc.

Corollary 2.4. *By considering the set of complex structure on $\mathbb{R}^2$, we get hyperboloid, upper half plane and the Poincaré disc as hyperbolic model, with their canonical metric.*

2.4 Quotients of the three simply connected Riemann Surface

In this subsection we discuss the **classification of Riemann surfaces**.

Given S a Riemann surface, therefore S is biholomorphic to $\tilde{S}/\pi_1(S)$, where $\tilde{S}$ is the universal cover of S and $\pi_1(S)$ is a discrete group whose actions on $\tilde{S}$ is biholomorphisms.

This action is properly discontinuous without fixed points. $\tilde{S}$ is indeed a connected and simply connected Riemann surface and $\tilde{S} \xrightarrow{\pi} S$ is holomorphic.

Since $\tilde{S}$ is simply connected we can apply the uniformization theorem. There is $\varphi : \tilde{S} \to U$ biholomorphism where U is either $\mathbb{P}^1, \mathbb{C}$ or D.

Case 1 $U = \mathbb{P}^1$ (when the universal cover is compact).

We have that $S = \mathbb{P}^1/\pi_1(S)$, with $\pi_1(S)$ the fundamental group of S that acts on $\mathbb{P}^1$ without fixed points by biholomorphism.

We know that $\operatorname{Aut}(\mathbb{P}^1) = \mathrm{PSL}(2, \mathbb{C})$, where every $\gamma = \frac{az+b}{cz+d} \in \mathrm{PSL}(2, \mathbb{C})$ admits at least one fixed point (since there is always a "proper line", the matrix can be triangulated). Then $\pi_1(S) = \{\mathrm{id}\}$, therefore $S = \tilde{S} = \mathbb{P}^1$. The unique Riemann surface that is covered by $\mathbb{P}^1$ is $\mathbb{P}^1$.

Corollary 2.5. *The only Riemann surface which is covered by $\mathbb{P}^1$ is itself, due to there is no biholomorphism map on $\mathbb{P}^1$ without fixed point.*

Case 2 $U = \mathbb{C}$.

Now S is holomorphic to $\mathbb{C}/\pi_1(S)$ where $\pi_1(S)$ acts on $\mathbb{C}$ by transformations of covering that are biholomorphisms. Recall that

$$\operatorname{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

If $a \neq 1$, the transformation $z \mapsto az + b$ always admits a fixed point in $\mathbb{R}$: $az + b = z \Rightarrow z = \frac{b}{1-a}$. Then $\pi_1(S) \subset \{z \mapsto z + b : b \in \mathbb{C}\}$. So $\pi_1(S)$ is a discrete subgroup of the translation group.

We have only 3 cases for $\pi_1(S)$ (if we have three translations that are rationally independent, we can prove that there are some actions converging to id, which contradicting proper discontinuity.)

(i) $\pi_1(S) \cong \mathbb{Z}$ generated by a translation $z \mapsto z + \omega$, in which case we have $\mathbb{C}/(z \mapsto z + \omega) \cong \mathbb{C}^*$ by $\exp\left(2i\pi \frac{z}{\omega}\right)$.

(ii) $\pi_1(S) \cong \mathbb{Z}^2$ generated by 2 translations $z \mapsto \omega_1$, $z \mapsto \omega_2$ with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$.
S is an elliptic curve $\mathbb{C}/\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, in particular S is biholomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.
$\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is biholomorphic to $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ with $\operatorname{Im}\tau > 0$. Moreover, $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ is biholomorphic to $\mathbb{C}/\mathbb{Z} \oplus \tau'\mathbb{Z}$ iff $\exists g \in \mathrm{PSL}(2, \mathbb{Z})$ such that $g\tau = \tau'$.

(iii) $\pi_1(S) = \{\mathrm{id}\}$. Then $S \cong \mathbb{C}$.

Corollary 2.6. *If $\tilde{S} \cong \mathbb{C}$ then $\pi_1(S)$ acts on $\mathbb{C}$ by translation. $S = \mathbb{C}/\pi_1(S)$ admits a flat Riemannian metric by $|dz|^2 = dx^2 + dy^2$ (covariant by translation) and it is compatible with the complex structure.*

Case 3 $U = D$. Hyperbolic geometry.

$\tilde{S}$ is holomorphic to $\mathbb{D}$, also to $\mathbb{H}$. In this case $S \cong \tilde{S}/\pi_1(S)$ where $\pi_1(S)$ is a discrete subgroup that acts on $\tilde{S}$ by biholomorphisms and the action is proper and discontinuous. Therefore $\pi_1(S)$ preserves the hyperbolic metric of S. This hyperbolic metric, induces a Riemannian metric on S, compatible with the complex structure with negative constant curvature.

Corollary 2.7. *If S is a (complete?) Riemann surface, then S admits a complete Riemannian metric of constant curvature compatible with the complex structure.*

More precisely, this metric h is a positive constant curvature if $\tilde{S} \cong \mathbb{P}^1$, 0 if $\tilde{S} \cong \mathbb{C}$ and negative if $\tilde{S} \cong D$.

Corollary 2.8. *If S is an orientable surface, any Riemannian metric on S is equivalent to a complete Riemann metric with constant curvature.*

In the case where $\tilde{S} = \mathbb{P}^1$ we saw that $\pi_1(S) = \{1\}$. When $\tilde{S} = \mathbb{C}$, we saw that $\pi_1(S) = \mathbb{Z}$ ($S = \mathbb{C}^*$) or $\pi_1(S) = \mathbb{Z}^2$ ($S = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$).

Results on hyperbolic case

From the discussion above, we've seen that the structure for the formal two cases is very easy, and our rich structure is in the last case.

Now what can we say about $\pi_1(S)$ when $\tilde{S} = \mathbb{D}$?

For a Riemann surface that is compact if genus $g \geq 2$, we know that $\pi_1(S)$ is not abelian (a topological result), then $\tilde{S} = D$.

Corollary 2.9. *Any compact Riemann surface of genus $g \geq 2$, is covered by D. In particular, admits a complete Riemannian metric of curvature -1.*

If $\tilde{S} = D$, $S = \tilde{S}/\pi_1(S)$, and $\pi_1(S)$ acts by biholomorphisms on $\tilde{S} = D$ preserving the hyperbolic metric and without fixed point.

Remark 2.9. *Any action of a discrete group that preserves a metric is proper and discontinuous.*

How can we know that a discrete subgroup $\Gamma = \pi_1(S) \leq \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{H})$ acts with a compact quotient and without fixed points?

Proposition 2.5. $\Gamma = \pi_1(S) \leq \mathrm{Aut}(\mathbb{H})$ *acts without fixed point iff* Γ *is without torsion (If $\gamma \in \Gamma$, $n \geq 1$ such that $\gamma^n = \mathrm{id}$ then $\gamma = \mathrm{id}$).* Γ *acts on* $\mathbb{H}$ *with a compact quotient iff* $\mathrm{PSL}(2, \mathbb{R})/\Gamma$ *is compact.*

Proof. First we say that $\mathbb{H} = \mathrm{PSL}(2, \mathbb{R})/\operatorname{Stab}(i)$. Moreover, for matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R})$, it fix i when $\frac{ai+b}{ci+d} = i$, that is,

$$a = d, b = -c, \text{with } a^2 + b^2 = 1 \Rightarrow \operatorname{Stab}(i) \cong \mathbb{S}^1.$$

Thus

$$\mathbb{H} = \mathrm{PSL}(2, \mathbb{R})/\operatorname{Stab}(i) \cong \mathrm{PSL}(2, \mathbb{R})/\mathbb{S}^1,$$

and the map $\mathrm{PSL}(2, \mathbb{R}) \to \mathbb{H}$ is a fiberation whose fiber as a circle. We identify this fiberation with the unit tangent bundle (tangent vectors with norm 1) for the hyperbolic metric.

Since the stabiliser is a rotation, we say $\mathbb{H}$ is a symmetric space, that is for any $x, y \in \mathbb{H}$, $v_1 \in T_x\mathbb{H}$ and $v_2 \in T_y\mathbb{H}$, with $|v_1| = 1$ and $|v_2| = 1$, then there is $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $gx = y$ and $dg_x v_1 = v_2$. This shows that the action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathrm{PSL}(2, \mathbb{R})$ is transitive.

$\mathrm{PSL}(2, \mathbb{R})/\Gamma$ is the unit tangent bundle of $\mathbb{H}/\Gamma$, hence $\mathrm{PSL}(2, \mathbb{R})/\Gamma$ is compact iff $\mathbb{H}/\Gamma$ is compact.

If Γ acts without fixed point, then obviously it has no torsion. Let's assume that Γ is without torsion, we shall show that Γ acts without fixed points in $\mathbb{H}$. In fact, if γ admits a fixed point, i.e. there is $x_0 \in \mathbb{H}$, $\gamma \cdot x_0 = x_0$, then $\forall n \in \mathbb{Z}, \gamma^n \cdot x_0 = x_0$.

Let's consider in the model $\mathbb{D}$ and assume $x_0 = 0$, we must have $\gamma^{-n} \in e^{i\theta}, \forall n \in \mathbb{Z}$. Since $\{\gamma^n : n \in \mathbb{Z}\}$ must be a discrete subgroup of $\mathbb{S}^1$, hence it is a finite set, then $\exists N \in \mathbb{N}$ such that $\gamma^N = 1$. Then γ is an element of torsion. ☐

Remark 2.10. 2 *quotient spaces of* $\mathbb{D}$: $\mathbb{D}/\Gamma_1, \mathbb{D}/\Gamma_2$ *with* Γ_1 *and* Γ_2 *discrete subgroups without torsion are biholomorphic iff* $\exists \varphi \in \mathrm{PSL}(2, \mathbb{R})$ *such that* $\varphi \circ \Gamma_1 \circ \varphi^{-1} = \Gamma_2$.

Proof. A biholomorphism from $\mathbb{D}/\Gamma_1$ to $\mathbb{D}/\Gamma_2$ lifts to a biholomorphism of the universal cover $\mathbb{D}$ with corresponding to an element of $\mathrm{PSL}(2, \mathbb{R})$ that conjugate Γ_1 and Γ_2. ☐

2.5 Some statements and the proof of uniformization theorem I

Theorem 2.4 (Gauss, Row-Liditenstewi)**.** *S is a surface and $j \in H^0(\mathrm{End}(TS))$ is an almost complex structure, then $\forall s \in S$, there is $\varphi : (v, s) \mapsto (\varphi(v) \in \mathbb{C}, 0)$ a local diffeomorphism between an open neighborhood U of s in S and $\varphi(U)$ an open neighborhood of 0 in $\mathbb{C}$ such that $\varphi(s) = 0$ and $d\varphi(j \cdot v) = id\varphi(v)$, $\forall v \in TU$.*

Remark 2.11. *This is a theorem of local integrability of almost complex structures in C^ω-case that we proved in Theorem 2.1.*

Recall that on a surface S, if S is endowed with an orientation, the almost complex structure j is given by a Riemannnian metric g (and $\lambda \cdot g$ with $\lambda : S \to \mathbb{R}_+$ defines the same j).

Theorem 2.5 (Poincaré-Rorbe)**.** *Let j be an almost complex structure on a simply connected surface $\tilde{S}$. Then there is a global diffeomorphism $\varphi : \tilde{S} \to M$, where M is either $\mathbb{P}^1, \mathbb{C}, D$, such that $d\varphi(j \cdot v) = id\varphi(v)$, $\forall v \in T\tilde{S}$.*

Recall that the space of complex structure on $\mathbb{R}^2$ = Hyperbolic space. If g is a quadratic form on $\mathbb{R}^2 \cong \mathbb{C}$,

- $g = ax^2 + 2bxy + cy^2$, $ac - b^2 > 0, a, c > 0$. This gives the hyperboloid model $\det \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -a^2 - bc = 1$. $g = a(x + \rho y)(x + \overline{\rho} y)$.
- $g = a \cdot |x + \rho y|^2, a > 0$. $\mathrm{Im}\, \rho > 0$. ρ gives the model of the upper-half plane.
- $g = \lambda |z + \Phi \overline{z}|^2$, $\lambda > 0$, $|\Phi| < 1$. Φ gives the model of the disc.

If $U \subset \mathbb{C}$ is an open set in $\mathbb{C}$, any almost complex structure on U is defined by the conformal class of a Riemannian metric $g = \lambda |dz + \Phi d\overline{z}|^2$ with $\Phi : U \to \mathbb{D}$.

Let $\omega = dz + \Phi d\overline{z} \in \Omega^1(U, \mathbb{C})$, it is a differential form of degree 1 which defines j by the formula $\omega(j \cdot v) = i\omega(v), \forall v \in TU$. In the case $\Phi = 0$, the ω gives the standard complex structure given the inclusion $U \subset \mathbb{C}$. By direct calculation, we have

$$j \frac{\partial}{\partial z} = i \frac{1 + |\Phi|^2}{1 - |\Phi|^2} \frac{\partial}{\partial z} + \frac{-2i\overline{\Phi}}{1 - |\Phi|^2} \frac{\partial}{\partial \overline{z}}.$$

Remark 2.12. *If $f : U \to \mathbb{C}$ is a C^∞-function, ω and $f\omega$ define the same almost complex structure.*

In order to prove the local integrability of j, one should find a local diffeomorphism $\Psi : V \subset U \to \Psi(V)$ in $\mathbb{C}$ such that $f \cdot \omega = d\Psi$. Indeed, in this case $d\Psi(j \cdot v) = f\omega(j \cdot v) = if\omega(v) = id\Psi(v)$. By Poincaré lemma, we just need to find f such that $f \cdot \omega$ is closed.

$\exists f, \Psi : U \to \mathbb{C}$ such that $f\omega = d\Psi$ in a neighborhood of a given point iff there exists $f : U \to \mathbb{C}$ such that $d(f\omega) \equiv 0$.

$$f\omega = f(dz + \Phi d\overline{z}) = f dz + (f\Phi) d\overline{z}.$$

Then $d(f\omega) = 0$ iff
$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial (f\Phi)}{\partial z},$$
this is called **Beltrami equation**. Moreover, Ψ is a local diffeomorphism iff f does not vanish.

Theorem 2.6 (**Isothermal coordinate theorem**). *Let $U \subset \mathbb{C}$ and an almost complex structure on U given by a map $\Phi : U \to D$ (j is define by the condition that $dz + \Phi d\overline{z} \in \Omega^1(U, \mathbb{C})$ is $\mathbb{C}$-linear). We can find a coordinate $\Psi : V \to \Psi(V)$ with $V \subset U \subset \mathbb{C}$ such that $d\Psi(j \cdot v) = id\Psi(v)$. It is equivalent to find a solution f of Beltrami equation $\frac{\partial f}{\partial \overline{z}} = \frac{\partial (f\Phi)}{\partial z}$ defined on V and such that $f(v) \neq 0$ for all $v \in V$.*

Remark 2.13. *To make $dz + \Phi d\overline{z}$ preserve the orientation, we need $|\Phi| < 1$.*

Lemma 2.2 (**Technical lemma**). *Let $\nu(z, t) : \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1] \to D$ be a smooth function such that $\nu(z, 0) \equiv 0$, then there exists a smooth function $f : \mathbb{R}^2 \times [0, 1]$ such that $\frac{\partial f}{\partial \overline{z}} = \frac{\partial f\nu}{\partial z}$ and $f(z, t)$ is not identically zero for any $t \in [0, 1]$ and $f(z, 0) \equiv 1$.*

Remark 2.14. *The method is for any unknown $\Phi : \mathbb{R}^2/\mathbb{Z}^2 \to D$, consider $\nu = t\Phi$.*

For any $t \in [0, 1]$ the almost complex structure is given by $\omega_t = dz + \nu(z, t)d\overline{z}$.

Lemma 2.3 (**Strong technical lemma**). *Moreover, in technical lemma we have $f(z, t)$ does not vanish for any t.*

Logic:

Technical lemma $\Rightarrow$ Isothermal coordinate theorem.

$$\begin{cases} \textbf{Technical lemma} \\ \textbf{Isothermal coordinate theorem} \end{cases} \Rightarrow \textbf{Strong Technical lemma.}$$

Strong Technical lemma $\Rightarrow$ Uniformization theorem.

Corollary 2.10 (Corollary of the strong technical lemma). *Let j any almost complex structure on $\mathbb{R}^2/\mathbb{Z}^2$. Then there exists a global diffeomorphism $\Psi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}/\Lambda$, with Λ some lattice in $\mathbb{C}$ such that $d\Psi(j \cdot v) = i \cdot d\Psi(v)$, for any $v \in T(\mathbb{R}^2/\mathbb{Z}^2)$.*

Proof. Let j be the almost complex structure and we can consider by taking $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{C}/\mathbb{Z}^2$ that j is given by a $\nu : \mathbb{C}/\mathbb{Z}^2 \to D$ such that $dz + \nu d\overline{z}$ is $j-\mathbb{C}$-linear. We consider the path of almost complex structure given by $t\nu(z)$, $\forall t \in [0, 1]$. This relates our j to the standard complex structure on $\mathbb{C}/\mathbb{Z}^2$ given by $\nu \equiv 0$ and $\omega = dz$.

By the strong technical lemma, there exists $f(z, t)$ with $z \in \mathbb{C}/\mathbb{Z}^2$ and $t \in [0, 1]$ such that $\frac{\partial f}{\partial \overline{z}} = \frac{\partial (f \cdot \nu_t)}{\partial z}$ for any $t \in [0, 1]$. Moreover, $f(\cdot, t)$ does not vanish for any $t \in [0, 1]$.

$\forall t \in [0, 1]$, $f(z, t) \cdot \omega$ where $\omega = dz + t\nu d\overline{z}$ is closed. In particular, for $t = 1$, $d(f(z, 1) \cdot \omega)$ is closed. This implies first that locally $\exists \Psi : U \subset \mathbb{C}/\mathbb{Z}^2 \to \Psi(V) \subset \mathbb{C}$ such that $d\Psi =$

$f(z,1)\cdot\omega$, meanings that our j is locally integrable (we have proved isothermal coordinate theorem for j).

Pull back the closed form $f(z,1)\cdot\omega$ to $\mathbb{R}^2\cong\mathbb{C}$, and denote the pull-back by $\tilde{\omega} = f(z,1)\cdot\omega$. There will be $\Psi:\mathbb{C}\to\mathbb{C}$ such that $\tilde{\omega}=f(z,1)\omega=d\Psi$, meaning that Ψ is a diffeomorphism (since $f(z,1)$ does not vanish, $d\Psi$ does not vanish) which conjugate the pull-back of j on $\mathbb{C}$ to $\times i$ on $\Psi(\mathbb{C})$. We will show now that $\Psi(\mathbb{C})=\mathbb{C}$. Ψ sends $\tilde{\omega}$ on dz (that is $\Psi^*(dz)=d\Psi=\tilde{\omega}$).

Ψ is an isometry in between $|\tilde{\omega}|^2$ and $|dz|^2=dx^2+dy^2$. Riemannian metrics on compact manifolds are complete, then $|f(z,1)\omega|^2$ is complete on $\mathbb{C}/\mathbb{Z}^2$, then $|\tilde{\omega}|^2$ is complete on $\mathbb{C}$, hence Ψ is a cover and since the target is simply connected, Ψ is a diffeomorphism between $\mathbb{C}$ and $\mathbb{C}$. This implies that ψ descends on a biholomorphism between $(\mathbb{C}/\mathbb{Z}^2,j)$ and $\mathbb{C}/\Lambda$, for some lattice Λ.

Notice that $f(z,1)\omega$ is a holomorphic form on $(\mathbb{C}/\mathbb{Z}^2,j)$ since it is $\mathbb{C}$-linear and closed. We proved that a Riemann surface with a non vanishing holomorphic form is $\mathbb{C}/\Lambda$.

A different point view, $f(z,1)\omega$ holomorphic gives a non vanishing holomorphic vector field X on S. Then $\varphi'(t)=X(\varphi(t))$ gives $\mathbb{C}/\operatorname{Stab}\to(\mathbb{C}/\mathbb{Z}^2,j)$ diffeomorphism. □

Corollary 2.11. *Let $\mathbb{T}=\mathbb{R}^2/\mathbb{Z}^2$ endowed with some Riemannian metric g. Then there is a flat metric on $\mathbb{R}^2/\mathbb{Z}^2$conformal to g.*

Proof. In the previous proof we showed that there exists a conformal diffeomorphism between $(\mathbb{R}^2/\mathbb{Z}^2,g)$ and $(\mathbb{C}/\Lambda,|dz|^2)$. $\Psi:\mathbb{R}^2/\mathbb{Z}^2\to\mathbb{C}/\Lambda$ is such that $\Psi^*(|dz|^2)=\lambda g$. □

The previous corollary proves isothermal coordinate theorem.

Let $0\in U\subset\mathbb{C}$ endowed with an almost complex structure given by $\nu:U\to D$. Restrict ν to a $D(O,\varepsilon)$ such that $|\nu|_{\infty,\overline{D(O,\varepsilon)}}\le\delta<1$. Extend ν a smooth function being 0 on $U\setminus D(O,2\varepsilon)$. Extend ν as a bi-periodic function defined of $\mathbb{R}^2/\mathbb{Z}^2$. Applying the previous corollary to $\mathbb{R}^2/\mathbb{Z}^2$ with the bi-periodic almost complex structure. It was proved that this is conjugated to the standard $\mathbb{C}/\Lambda$. In particular, ν on $D(O,\varepsilon)$ is conjugated to the standard complex structure on same disk $\subset\mathbb{C}$.

How to deduce uniformization theorem from the uniformization of almost complex structures on the torus?

We assume that the topological classification is known, meaning that the universal cover of a surface is diffeomorphic to $\mathbb{P}^1\cong\mathbb{S}^2$ or to $\mathbb{R}^2$. We reduce the problem to the case where the universal cover is diffeomorphic to $\mathbb{R}^2$.

Indeed, assume that the universal cover is $\mathbb{S}^2$. We remove a point p, now the complex structure on $\mathbb{S}^2\setminus\{p\}$ (diffeomorphic to $\mathbb{R}^2$) is either D or $\mathbb{C}$

$$\Psi:(\mathbb{S}^2\setminus\{p\},j)\xrightarrow{\text{biholomorphism}}D\text{ or }\mathbb{C}.$$

The target of Ψ cannot be D, because in this case Ψ is bounded and by Riemann moving singular theorem, Ψ extends to $\tilde{\Psi}:\mathbb{S}^2\to D$, a contradiction with maximal principle.

Then $\Psi : \mathbb{S}^2 \setminus \{p\} \to \mathbb{C} \subset \mathbb{P}^1$ which is a biholomorphism. Notice that Ψ is a homeomorphism, so it sends s a neighborhood of P in $\mathbb{S}^2$ to a neighborhood of ∞ in $\mathbb{P}^1$. Then coordinate of $\mathbb{P}^1$ at the infty is $\frac{1}{z}$, then $\frac{1}{\Psi}$ is bounded in the neighborhood of P. Ψ extends to a holomorphic neighborhood form $\mathbb{S}^2$ to $\mathbb{P}^1$ which has a non zero derivative in P since Ψ is injective.

Hence we can only consider the case of $\mathbb{R}^2$.

Let us now consider the case where $\mathbb{R}^2$ is endowed with an almost complex structure $\gamma : \mathbb{R}^2 \cong \mathbb{C} \to D(O,1)$. We want to show that there exists a biholomorphism $\Psi : (\mathbb{R}^2, \nu)$ and D or $\mathbb{C}$.

Consider an exhaustion of $\mathbb{R}^2$ by relatively compact sets

$$S_1 \subset S_2 \subset S_3 \subset \cdots \subset \mathbb{R}^2, \overline{S_n} \subset S_{n+1}, \overline{S_n} \text{ compact}, \bigcup_{n\in\mathbb{N}} S_n = \mathbb{R}^2.$$

We can choose S_n to be simply connected. Consider $\nu|_{S_n} = \nu_n$ defines a bounded almost complex structure on S_n (there is $\delta_n < 1$ s.t. $|\nu|_{\infty,\overline{S_n}} \leq \delta_n < 1$).

As in the proof of isothermal coordinate theorem, we extend the complex structure (S_n, ν_n) to a bi-periodic complex structure on some torus $\mathbb{R}^2/\Lambda$.

Our theorem of classification of almost complex structure $\mathbb{R}^2/\Lambda$, there is $\Psi : (S_n, \nu_n) \to (\Psi(S_n), \times i)$, and Ψ is a biholomorphism with $\Psi(S_n)$ a simply connected open set in $\mathbb{C}$.

By the Riemann theorem (see homework), $\Psi(S_n)$ is biholomorphic to the unitary disc D. So there exists a biholomorphism $\Psi_n : (S_n, j) \to D$ (with standard complex structure). Moreover, we can assume that $O \in S_n$, $\Psi_n()) = O \in D$ and moreover we multiply Ψ_n by $\lambda_n = \frac{1}{\Psi'(O)}$ such that $\tilde{\Psi}_n = \lambda_n \cdot \Psi_n : (S_n, j) \to \lambda_n D$ is such that $\tilde{\Psi}(O) = O$ and $\tilde{\Psi}'_n(O) = 1$.

Fix $k \in \mathbb{N}$ and look for $\tilde{\Psi}_n \circ \Psi_k^{-1} : D \to \lambda_n D$, $(\tilde{\Psi}_n \circ \Psi_k^{-1})(O) = O$ and $\tilde{\Psi}_n \circ \Psi_k^{-1}$ have all the same derivative at O. Then it is a normal family in the Montel sense and we can have a subsequence which converges. We find the uniformization map by a diagonal extraction: the limit will be a holomorphic diffeomorhism from $(\mathbb{R}^2, j)$ to an open set in $\mathbb{C}$ (which is simply connected). By Riemann theorem. it is either D or $\mathbb{C}$.

Diagonal extraction: $(\tilde{\Psi}_n)_{n\geq 1}$ is a normal family on $S_k \subset S_{k+1} \subset \cdots$.

Proposition 2.6. *Technical lemma + isothermal coordinate theorem, imply strong technical lemma.*

Proof. Let us consider $f(z,t)$ the solution fo the Beltrami equation. We show that $\{t \in [0,1] : f(z,t)$ vanishes somewhere on $\mathbb{R}^2/\mathbb{Z}^2\}$ is a closed and open set in $[0,1]$. Since $f(z,0) \equiv 1$, then this set is not $[0,1]$, it is $\varnothing$.

Let us first show that this set is closed. Let $t_k \in [0,1]$ such that there is $z_k \in \mathbb{R}^2/\mathbb{Z}^2$ such that $f(z_k, t_k) = 0$ and $\lim_{k\to\infty} t_k = t_\infty \in [0,1]$. Then $(z_k) \subset \mathbb{R}^2/\mathbb{Z}^2$, then there is $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\lim_{k\to\infty} z_{t_{\sigma(k)}} = z_\infty \in \mathbb{R}^2/\mathbb{Z}^2$. Then $f(z_\infty, t_\infty) = 0$, hence t_∞ is in the set which is closed. $\square$

Now we prove it is also an open set. Let to be such that $f(z_0, t_0) = 0$. We can assume (by changing z_0) that f is not identically zero in the neighborhood of z_0. By isothermal coordinate theorem, we know that the complex structure $\nu(z,t_0)$ is integrable. In local holomorphic coordinates $f(z,\nu_o)(dz+\nu(z,t_0)d\bar{z})$ is a holomorphic form (it is closed and $\mathbb{C}$-linear). In local coordinate w, is of the form $w^n h(w)dw$, h holomorphic $h(0)\neq 0$ and $w(z_0)=0$. When we change $t_0\to t$, the order of vanishing of this section is still n, hence $f(z,t)$ vanishes for t closed to t_0.

Exercise 2.3. *Take a $2:1$ manifold cover of $\mathbb{P}^1(\mathbb{C})$ above for 4 points: $\{a,b,c,\infty\}$, then the Riemann surface is biholomorphic to $\mathbb{C}/\Lambda$.*

Proof. $\omega=\frac{dz}{\sqrt{(z-a)(z-b)(z-c)}}$ pull-back to a holomorphic form which does not vanish.

The coordinate above the point a will be v such that $z-a=v^2$. To test what happens at $v=0$ $(z=a)$. We have

$$\frac{dz}{\sqrt{(z-a)}}=\frac{d(v^2)}{v}=2v\frac{dv}{v}=2dv,$$

so on the cover the holomorphic 1-form do not have singularity above $z=a$. At the ∞ in $\mathbb{P}^1$, $z=\frac{1}{u^2}$, and

$$\frac{dz}{\sqrt{z^3}}=\frac{-2u^{-3}}{u^{-3}}=-2du.$$

$\square$

Remark 2.15. *For a Riemann surface which has a hole, if there is ω holomorphic 1-form. We can find γ_1 and γ_2 two closed path such that $\int_{\gamma_1}\omega=a\in\mathbb{C}\backslash\{0\}$ and $\int_{\gamma_2}\omega=b\in\mathbb{C}\backslash\{0\}$. Then $\int_0^z\omega$ maps S to $\mathbb{C}/\Sigma$.*

2.6 The proof of uniformization theorem II

Lemma 2.4. *Let $\gamma(z,t):T^2\times[0,1]\to\mathbb{C}$, $|\nu(z,t)|<1$ and ν is smooth (this defines a family of complex structures on T^2 such that $dz+\nu(z,t)d\bar{z}$ is $\mathbb{C}$-linear). We assume that $\nu(z,0)\equiv 0$. Then there exists a solution $f(z,t)$ to Beltrami equation $\frac{\partial}{\partial\bar{z}}f=\frac{\partial}{\partial z}(f\nu)$, $\forall z\in T^2$ and $t\in[0,1]$ s.t. $f(z,0)=1$, and $f(z,t)$ is not constant 0 in z, $\forall t\in[0,1]$.*

We will see later that $f(z,t)$ does not vanish for any $t\in[0,1]$.

Proof. Since $f(z,0)\equiv 1$, any function $f(z,t)$ satisfying

$$\partial_{\bar{z}}\dot{f}-(\partial_z\circ\nu)\dot{f}=(\partial_z\circ\dot{\nu})f,\quad f(z,0)\equiv 1,$$

is a solution of Beltrami equation.

Here $\dot{f}$ and $\dot{\nu}$ are partial derivative with respect to t and operations $\partial_z\circ\nu$ are multiplication by ν followed by $\frac{\partial}{\partial z}$.

$T^2, z=x_1+ix_2$, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial\bar{z}}$ admit eigenvectors which are $l_n(x)=e^{i(n_1\cdot x_1+n_2\cdot x_2)}$, where $n=(n_1,n_2)\in\mathbb{Z}$, with eigenvalues

$$\lambda_n=\frac{1}{2}(in_1+n_2),\quad \lambda_n'=\frac{1}{2}(in_1-n_2).$$

Then $|\lambda_n|=|\lambda_n'|$ and $\lambda_n'=-\overline{\lambda_n}$.

Corollary 2.12. *There exists a unique unitary operation in $L^2(\mathbb{R}^2/\mathbb{Z}^2)$ such that $U \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \circ U = \frac{\partial}{\partial z}$.*

Proof of the corollary. Define U as the operator which admits $l_n(x)$ as eigenvectors with eigenvalues $\frac{\lambda}{\lambda'}$ (with modules 1).

Moreover, it commutes with partial derivatives and we can treat U as $\partial_{\bar{z}}^{-1} \circ \partial_z$. □

We will solve the equivalent equation

$$(\mathrm{id} - U \circ \nu)(\dot{f}) = (U \circ \dot{\nu})(f).$$

Then it's equivalent to solve the following ordinary differential equation in a Banach space

$$\dot{f} = (\mathrm{id} - U \circ \nu)^{-1}(U \circ \dot{\nu})(f),$$

with initial value $f(z, 1) \equiv 1$.

It is enough to show that the norm of the linear operators is uniformly bounded w.r.t. t. Then this will imply that there is a unique solution of the ODE, $f(z,t) \in L^2(T^2)$ s.t. $f(z,t)$ is trivial for any t.

Notice that $|\nu(z,t)| < 1$, then $\|\nu(z,t)\| \leq \delta < 1$ for any $z \in T^2$ and $t \in [0,1]$. Then $\|U \circ \nu\|_{L^2} \leq \delta < 1$, which gives that $\mathrm{id} - U \circ \nu$ is invertible and

$$(\mathrm{id} - U \circ \nu)^{-1} = \sum_{k=0}^{\infty} (U \circ \nu)^k \Rightarrow \|(\mathrm{id} - U \circ \nu)^{-1}\|_{L^2} \leq \sum_{k=0}^{\infty} \delta^k = \frac{1}{1-\delta}.$$

Also $|\dot{\nu}| < \delta'$ then $\|U \circ \dot{\nu}\|_{L^2} \leq \delta'$, hence

$$\|(\mathrm{id} - U \circ \nu)^{-1}(U \circ \dot{\nu})\|_{L^2} \leq \frac{\delta'}{1-\delta}.$$

Now we still have a problem to prove the $f \in L^2(L^2)$ is actually smooth. We can prove the operator is uniformly bounded w.r.t. the norm of the Banach space H^s. Since $\bigcap_{s \geq 0} H^s(T^2) = C^\infty(T^2)$, we say f is smooth.

Definition 2.5. *The Sobolev space*

$H^s(T^2) = \{f \in L^2(T^2) :$ *all partial derivatives of order* $\leq s$ *are in* L^2 *(in the sense of distributions)*$\}$

and it is also $\{f \in \sum_{n,m} \nu_{n,m} e^{i(nx+my)} : (1+n^2+m^2)^s u_{n,m} \in l^2\}$.

The final part is to prove that $f(z,t)$does not vanish.

It is enough to prove that isothermal coordinate theorem because in this case we have seen that technical lemma implies strong technical lemma.

Proposition 2.7. *The technical lemma implies isothermal coordinate theorem.*

Proof of the proposition. We can start with an almost complex structure defined on $0 \in U \subset \mathbb{C}$ and determined by $\mu : U \to D(0,1)$. we can assume that $\mu(0) = 0$ (this means that by a linear conjugacy $j(0) = j_0$). We can assume also that $\|\nu\|_{\infty, C^3}$ is small by rescaling by a homotheties $\lambda < 1$ and restricting the definition domain $((f(\lambda x)' = \lambda f'(\lambda x)$, by choosing λ small enough, the derivative can be small enough).

We extend smoothly a function μ on $\mathbb{R}^2/\mathbb{Z}^2$ and in order to have the same notations as in the technical lemma we denote by $\nu(z,t) = t{\cdot}\mu$. $\nu(z,0)$ is the standard complex structure and $\gamma(z,q)$ is our almost complex in a small neighborhood of 0 which was extended on $\mathbb{R}^2/\mathbb{Z}^2$.

There is a solution $f(z,t)$ of the Beltrami equation. In particular $f(z,1)(dz+\nu(z,1)d\overline{z}) = f(z,1)(dz + \mu d\overline{z})$ is closed.

In order to prove isothermal coordinate theorem, it is enough to prove that $f(0,1) \neq 0$. Because then, by continuity, in the neighborhood of 0, we have $f(x,1)$ does not vanish. Then $f(x,1)(dz + \mu d\overline{z}) = d\Psi$ and Ψ is a diffeomorphism in the neighborhood of 0. In the coordinate Ψ, the complex structure given by μ is the standard one.

Our $f(z,t)$ is solution of $\dot{f} = (\mathrm{id} - U \circ \nu)^{-1}(U \circ \dot{\nu})(f)$. Notice that for the initial value $(f(z,0) \equiv 1)$, our solution is also $f(z,t) = (1 - U \circ \nu)^{-1}(1)$. Indeed, $t = 0$, we verify that $f(z,0) \equiv 1 \equiv \mathrm{id}(1) \equiv 1$, using $d(A^{-1})(H) = -A^{-1}HA^{-1}$, we have

$$\begin{aligned}\frac{\partial f(z,t)}{\partial t} &= (\mathrm{id} - U \circ \nu)^{-1} \circ (U \circ \dot{\nu}) \circ (\mathrm{id} - U \circ \nu)^{-1}(1) \\ &= (\mathrm{id} - U \circ \nu)^{-1} \circ (U \circ \dot{\nu})(f(z,t))\end{aligned}$$

We prove that for $\|u\|_{\infty, C^3}$ small enough, $(\mathrm{id} - t \circ U \circ \mu)^{-1}(1)$ is closed to 1 in the H^3-topology. If two functions are close in H^3-topology, they are closed in C^0-topology, hence $f(z,t)$ does not vanish.

$$\begin{aligned} C^3(T^2) &\xrightarrow{C^0} H^3(T^2); \\ u &\mapsto (\mathrm{id} - t \circ U \circ \mu)^{-1}(1). \end{aligned}$$

□

□

Definition 2.6 (Almost complex structures in higher dimension)**.** *Let X be a real manifold and $j \in H^0(X, \mathrm{End}(TX))$ s.t. $j^2 = -\mathrm{id}$. This implies that for each $x \in X$, T_xX admits a complex structure given by $\forall v \in T_xX$, $i \cdot v = j(x) \cdot v$.*

This implies $\dim X$ *is even, because* $\det(j^2) = (-1)^{\dim X}$.

TX is then a complex vector bundle, but not always a holomorphic vector bundle.

$$TX \subset TX_{\mathbb{C}} = TX \otimes \mathbb{C}.$$

Here the local sections of $TX \otimes \mathbb{C}$ are $X + iY$, where X, Y are local section of TX.

If $\dim X = 2n$, then $TX \otimes \mathbb{C}$ has complex rank $2n$. j extends as $H^0(X, \mathrm{End}(TX \otimes \mathbb{C}))$, we still have $j^2 = -\mathrm{id}$. $j(x) \in \mathrm{End}(T_x X \otimes \mathbb{C})$ has 2 eigenvalues which are $\pm i$. Consider $T_x^{(1,0)} X$ the eigenspace of the eigenvalue i and $T_x^{(0,1)} X$ the the eigenspace of the eigenvalue $-i$. Then $TX \otimes \mathbb{C} = T_x^{(1,0)} X \oplus T_x^{(0,1)} X$.

Example 2.2 (Standard example)**.** *Let X be a complex manifold, then $\times i$ is well-defined in local holomorphic coordinates and gives a $j \in H^0(X, \mathrm{End}(TX))$ which does not depend on the local coordinates and it is defined intrinsically on X. In holomorphic local coordinates*

$$(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \cdots, z_n = x_n + iy_n),$$
$$j(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial y_1}, \quad j(\frac{\partial}{\partial y_1}) = -\frac{\partial}{\partial x_n}.$$

In this case, $TX \otimes \mathbb{C} = T^{(1,0)} X \oplus T^{(0,1)} X$, where $T^{(1,0)} X$ is the subspace where j acts by the eigenvalue i and it is generated in those local coordinates by

$$(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial x_n} - i\frac{\partial}{\partial y_n}) = (\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}).$$

$T^{(1,0)} X$ admits the structure of a holomorphic vector bundle isomorphic to the holomorphic tangent bundle of X.

Moreover, $T^{(0,1)} X$ is generated in local holomorphic coordinate

$$(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n}) = (\frac{\partial}{\partial \overline{z}_1}, \frac{\partial}{\partial \overline{z}_2}, \cdots, \frac{\partial}{\partial \overline{z}_n}) = \overline{T^{(1,0)} X}.$$

Question: If X is a real manifold of $\dim 2n$, and $j \in H^0(X, \mathrm{End}(TX))$ s.t. $j \circ j = -\mathrm{id}$. Does there exists local coordinates in X such that j reads in those coordinates as $\begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & \\ & \vdots & \\ & & \begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$.

Remark 2.16. *If such local coordinates do exists at each point of X, thus the transition maps have a differential which commutes with $\times i$, then the transition maps form a holomorphic atlas and X has the structure of a complex manifold. For which $T^{(1,0)} X$ is the holomorphic tangent bundle TX and $T^{(0,1)} X = \overline{T^{(1,0)} X} = \overline{TX}$.*

*More concretely, one wants to find in the neighborhood of any point $p \in X$, an open set $p \in U \subset X$ and $\psi : U \to \psi(U) \subset \mathbb{C}^n$ such that $d\psi \circ j = i \times d\psi$. In this case all (ψ, U) will form a holomorphic atlas of X. Such a j is called **integrable almost complex structure**.*

Integrability condition for almost complex structure:

Theorem 2.7 (Newlander-Nirenberg)**.** *Let X be a real manifold endowed with an almost complex structure j. Then j is integrable if and only if*

$$[T^{(0,1)} X, T^{(0,1)} X] \subset T^{(0,1)} X.$$

Since we have seen that $T^{(1,0)} X = \overline{T^{(0,1)} X}$, this condition is equivalent with $[T^{(1,0)} X, T^{(1,0)} X] \subset T^{(1,0)} X$

This condition is satisfied if X is a complex manifold because $T^{(0,1)}X$ is generated by vector fields $X + i \cdot jX$ and

$$[X_1 + ijX_1, X_2 + ijX_2] = \cdots = (2[X_1, Y_1]) + ij(2[X_1, Y_1]).$$

(Exercise, since j is constant matrix, we have $[jX, Y] = j[X, Y]$).

Proof of Newlander-Nirenberg theorem in the case where j is real-analytic. j can be extended as a $\hat{j} \in H^0(\hat{U}, \operatorname{End}(T\hat{U}) \cong \mathbb{C}^{2n})$ such that $\hat{j}^2 = -\operatorname{id}$.

Now define $E_{\mathbb{C}}$ as being the eigenspace of $(-i)$ in $\mathbb{C}^{2n}$, $\dim_{\mathbb{C}} E_{\mathbb{C}} = n$.

Steps of the proof.

1. Show that $[E_{\mathbb{C}}, E_{\mathbb{C}}] \subset E_{\mathbb{C}}$.

2. Prove Frobenius theorem saying that: if X is a complex manifold of dimension n, and $E \subset TX$ is a holomorphic sub-bundle of rank k, such that $[E, E] \subset E$, then for any $x \in X$, there exists an open neighborhood $x \in W \subset$ and a holomorphic submersion $\psi : W \to \psi(W) \subset \mathbb{C}^{n-k}$ s.t. $E_u = \operatorname{Ker} d\psi(u)$, for any $u \in W$. This means that locally E coincide with the tangent space of the fibers of a fibration.

3. Apply Frobenius theorem of $E_{\mathbb{C}}$ (of rank n) and find local submersion $\psi : \tilde{U} \subset \mathbb{C}^{2n} \to \mathbb{C}^n$ and prove that $\psi|_U : U \to \mathbb{C}^n$ is a diffeomorphism and $d\psi \circ j = i \circ d\psi$.

□

3 Sheaf theory

3.1 Dolbeault complex

M a complex manifold with $TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$.

Dual decomposition $\Omega_M \otimes \mathbb{C} = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$. The forms in $\Omega^{(1,0)}$ are locally $\sum_i f_i(z_1, \cdots, z_n)dz_i$, where f_i isa smooth function and forms in $\Omega^{(0,1}$ are locally $\sum_i g_i(z_1, \cdots, z_n)d\overline{z}_i$, where g_i is a smooth function.

$\Omega^k_M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}$, where sections of $\Omega^{p,q}$ are locally given by $i = (i_1, i_2, \cdots, i_p)$, $j = (j_1, j_2, \cdots, j_n)$,

$$\sum_{i,j} f_{ij} dz_i \wedge d\overline{z}_j,$$

where f_{ij} is a smooth function, $dz_i = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\overline{z}_j = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_p}$.

If $\alpha = \sum_{i,j} f_{ij} dz_i \wedge d\overline{z}_j \in \Omega^{p,q}$, then

$$d\alpha = \sum_{i,j} (df_{ij}) \wedge dz_i \wedge d\overline{z}_j \in \Omega^{p+1,q} \oplus \Omega^{p,q+1},$$

We will say that $d = \partial + \overline{\partial}$, where $\partial\alpha \in \Omega^{p+1,q}$ and $\overline{\partial}\alpha \in \Omega^{p,q+1}$.

In particular if $f \in \Omega^{0,0}$, then $df = \partial f + \overline{\partial} f$. Then f is holomorphic iff $\overline{\partial} f = 0$. We say $\alpha \in \Omega^{1,0}$ is holomorphic iff $\overline{\partial}\alpha = 0$.

Proposition 3.1. *Properties of operators $\partial, \overline{\partial}$:*

i) $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \partial\beta$.

ii) $\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \overline{\partial}\beta$.

iii) $\partial \circ \partial = 0$, $\overline{\partial} \circ \overline{\partial} = 0$, $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$.

De Rham complex $\Omega^k = \{\text{smooth forms of degree } k\}$

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} 0, \quad d \circ d = 0.$$

Dolbeault complex

$$0 \to \Omega^{0,0} \xrightarrow{d} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n} \xrightarrow{\overline{\partial}} 0, \quad \overline{\partial} \circ \overline{\partial} = 0.$$

For de Rham operator, we have Poincaré lemma: $\alpha \in \Omega^k(U)$ with $k \geq 1$, $d\alpha = 0$, then there is $\beta \in \Omega^{k-1}(V)$ s.t. $d\beta = \alpha$ for some $V \subset U$.

For Dolbeault operator, we have Poincaré lemma: $\alpha \in \Omega^{0,q}(U)$ with $q \geq 1$, $\overline{\partial}\alpha = 0$, then there is $\beta \in \Omega^{0,q-1}(V)$ s.t. $\overline{\partial}\beta = \alpha$ for some $V \subset U$.

Moreover, $\overline{\partial}$-operator extends to the following situation. Let M be a complex manifold and $E \to M$ a holomorphic vector bundle of rank n. We define $\Omega^{p,q}(E) = \Omega^{p,q}_M \otimes E$ and an operator

$$\overline{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E),$$

s.t. for a holomorphic trivialization $E|_U = U \times \mathbb{C}^n$ with local sections $(s_1, \cdots, s_r)$ of $\Omega^{p,q} \otimes E$, then we define

$$(\overline{\partial} s_1, \cdots, \overline{\partial} s_r) \in \Omega^{p,q+1}(U).$$

This does not depend on the holomorphic trivialization and gives an operator $\overline{\partial}$ s.t. $\overline{\partial} \circ \overline{\partial} = 0$, and we have

$$0 \to \Omega^{0,0}(E) \xrightarrow{\overline{\partial}} \Omega^{0,1}(E) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n}(E) \xrightarrow{\overline{\partial}} 0,$$

and the kernel of $\Omega^{0,0}(E) \xrightarrow{\overline{\partial}} \Omega^{0,1}(E)$ are the holomorphic sections of E.

3.2 Sheaves and sheaf cohomology

This is a tool to deal with gluing problem: going from local data to global data.

Example 3.1. *Let X be a Riemann surface and $P_1, \cdots, P_r, Q_1, \cdots, Q_l$ points on X and $n_1, \cdots, n_r \in \mathbb{N}$, $m_1, \cdots, m_l \in \mathbb{N}$. **Mittag-Leffler Question:** Does there exists on X meromorphic function which admits polos of order at most n_i at P_i and zeros of oder at least m_i at Q_i?*

Example 3.2. *Another question is that: Does vector bundles admit global sections?*

Let X be a Riemann surface (but the theory could be developed in more general context of topological spaces).

Definition 3.1. *A **pre-sheaf** $\mathcal{F}$ of abelian groups over X (or vector spaces) is the following data:*

*For each open set $U \subset X$, we have a group $(\mathcal{F}(U))$ (or a vector space) and for each pair of open sets $V \subset U \subset X$, we have a morphism (called the **restriction morphism**) $\rho_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$ with the properties: (i) $\mathcal{F}(\varnothing) = 0$, (ii)$\rho_U^U = \mathrm{id}$, (iii)$\rho_U^W = \rho_V^W \circ \rho_U^V$, for any $W \subset V \subset U \subset X$.*

*$\mathcal{F}(U)$ are called the **sections** of $\mathcal{F}$ over U. $s \in \mathcal{F}(U)$ is called a section of $\mathcal{F}$ over U. $s \in \mathcal{F}(X)$ is called a global section.*

Definition 3.2. *A pre-sheaf $\mathcal{F}$ is a **sheaf** iff for any open set $U \subset X$ and any covering of U by open sets U_i: $U = \bigcup\limits_i U_i$, we have that the following is an exact sequence*

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha:(\rho_U^{U_i})} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta:\rho_{U_i}^{U_i \cap U_j} - \rho_{U_j}^{U_i \cap U_j}} \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

Pre-sheaf definition: $\beta \circ \alpha = 0$; 1st axiom of sheaves: α injective; 2nd axiom of sheaves: $\operatorname{Im} \alpha \supset \operatorname{Ker} \beta$.

This means that if $U = \bigcup\limits_i U_\alpha$ and $s_\alpha \in \mathcal{F}(U_\alpha)$ s.t. $\rho_{U_\alpha}^{U_\alpha \cap U_\beta} s_\alpha = \rho_{U_\beta}^{U_\alpha \cap U_\beta} s_\beta$, for any $\alpha, \beta < 1$ s.t. $U_\alpha \cap U_\beta \neq \varnothing$, then there exists a unique $s \in \mathcal{F}(U)$ such that $\forall \alpha \in I, \rho_U^{U_\alpha} s = s_\alpha$.

Definition 3.3. *For $P \in X$ the **fibers** of $\mathcal{F}$ in P is $\mathcal{F}_P = \varinjlim_{U \ni p} \mathcal{F}(U)$. $\mathcal{F}_P$ is the space of germs of sections at P: $s \in \mathcal{F}(U_x)$ and $s' \in \mathcal{F}(V_x)$ with U_x, V_x being open sets and $x \in U_x \cap V_x$, we will say $s \sim s'$ if there is an open set $x \in W_x \subset U_x \cap V_x$ and $\rho_{U_x}^{W_x} s = \rho_{V_x}^{W_x} s'$. The space of equivalence classes is $\mathcal{F}_P$.*

Example 3.3. *X a Riemann surface and $\mathcal{O}$ the sheaf of holomorphic functions on X, $\mathcal{F}_p \cong \mathbb{C}[z]$ the space of convergent power series at 0.*

Example 3.4. *X a Riemann surface and $P \in X$, we define the **sky-scraper** sheaf in the following way:*

$$\mathcal{F}(U) = \begin{cases} \mathbb{C}, & P \in U \\ 0, & P \notin U \end{cases}, \quad \rho_U^V = \begin{cases} \mathrm{id}, & P \in V \\ 0, & P \notin V \end{cases}, \quad \mathcal{F}_Q = 0, \text{ if } Q \neq P, \quad \mathcal{F}_P = \mathbb{C}.$$

Example 3.5. *$\mathcal{E}$ the sheaf of smooth functions. $\mathcal{E}(U)$ is the vector space of smooth functions on U. $\mathcal{E}^1$ the sheaf of smooth 1-forms. $\mathcal{E}^k$ the sheaf of k-forms. $\mathcal{E}^{1,0}$ the sheaf of smooth $(1,0)$-forms and $\mathcal{E}^{0,1}$ the sheaf of smooth $(0,1)$-forms.*

Z^1 the sheaf of closed 1-forms and Ω the sheaf of holomorphic forms (type $(1,0)$ and closed). $\mathcal{M}$ the sheaf of meromorphic functions. $\mathcal{E}^*$ the sheaf of non-vanishing smooth functions. $\mathcal{C}^*$ the sheaf of non-vanishing continuous functions. $\mathbb{C}$ the sheaf of locally constant functions. $\mathbb{R}$ the sheaf of locally constant functions with real value. $\mathbb{Z}$ the sheaf of locally constant functions with integer value.

Definition 3.4 (Morphisms of sheaves)**.** *X a Riemann surface. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups (or vector spaces) over X. A morphism from $\mathcal{F}$ to $\mathcal{G}$ is the data:*

For any $U \subset X$ an open set, there exists a group homomorphism from $\mathcal{F}(U)$ into $\mathcal{G}(U)$, called f_U with the compatibility condition: if $V \subset U$ the following diagram should commute

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho_U^V"'] & \mathcal{G}(U) \arrow[d, "\rho_U^V"] \\ \mathcal{F}(V) \arrow[r, "f_V"] & \mathcal{G}(V) \end{tikzcd}.$$

Remark 3.1. *If $U = \bigcup_{i \in I} U_i$, f_U is completely determined by f_{U_i} if they are defined in compatible way.*

Example 3.6. *Plenty examples of morphisms of sheaves:*

$d : \mathcal{E} \to \mathcal{E}^1$, $d : \mathcal{E}^1 \to \mathcal{E}^2$.

$\partial : \mathcal{E} \to \mathcal{E}^{1,0}$, $\partial : \mathcal{E}^{0,1} \to \mathcal{E}^{1,1} = \mathcal{E}^2$.

Dolbeault operator $\overline{\partial} : \mathcal{E} \to \mathcal{E}^{0,1}$, $\overline{\partial} : \mathcal{E}^{1,0} \to \mathcal{E}^{1,1}$.

$\Delta = 2i\partial\overline{\partial} : \mathcal{E} \to \mathcal{E}^2$.

$\exp : \mathcal{O} \to \mathcal{O}^*$, $\exp : \mathcal{E} \to \mathcal{E}^*$, $\exp : \mathcal{C} \to \mathcal{C}^*$.

Inclusion morphisms: $\mathbb{R} \to \mathbb{C}$, $\mathcal{E} \to \mathcal{O}$...

Remark 3.2. *If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then f defines $f_x : \mathcal{F}_x \to \mathcal{G}_x, \forall x \in X$.*

Definition 3.5. *We say that the sequence of morphisms $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is **exact** if $\forall x \in X$, $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is an exact sequence.*

The criteria for exactness is the following: $\forall x \in X, \forall g_x \in \mathcal{G}_x$, $\beta_x g_x = 0 \iff \exists f_x \in \mathcal{F}_x, g_x = \alpha_x f_x$.

This is equivalent to: $\forall x \in X$, $\forall U \ni x$ open set, $\forall g \in \mathcal{G}(U)$ and $\exists x \in W \subset U$, $\beta_W g|_W = 0$ iff $\exists x \in V \subset W$ open set, $\exists f \in \mathcal{F}(V)$ s.t. $\alpha_V(f) = g|_V$.

It is also equivalent to: $\forall \Omega \subset X$ an open set, $\forall g \in \mathcal{G}(\Omega)$ with $\beta_\Omega g = 0$, with the covering $\Omega = \bigcup_{i \in I} U_i$, $\exists f_i \in \mathcal{F}(V_i)$ s.t. for any i, $\alpha_{U_i} f_i = \rho_\Omega^{U_i} g$.

Definition 3.6. *We say that $\alpha : \mathcal{F} \to \mathcal{G}$ is surjective if $\mathcal{F} \to \mathcal{G} \to 0$ is exact. $\beta : \mathcal{G} \to \mathcal{H}$ is injective if $0 \to \mathcal{G} \to \mathcal{H}$ is exact.*

Remark 3.3. *β is injective $\iff \forall x \in X, \beta_x$ is injective $\iff \forall U \subset X$ open set β_U is injective.*

However, α is surjective cannot imply α_U is surjective! Here is a counter-example

$$0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} Z^1 \to 0$$

is exact due to Poincaré lemma but usually $d(\mathcal{E}) \subsetneqq Z^1$ (surjectivity of morphism between sheaf is a local property).

Proposition 3.2.

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is exact,

a) $\forall U \subset X$ open set, α_U is injective (α identifies sections of $\mathcal{F}$ with sections in $\mathcal{G}$).

b) $\forall U \subset X$ open set, $\forall g \in \mathcal{G}(U)$, we have that $\beta_U g = 0$ iff there exists $f \in \mathcal{F}(U)$ s.t. $\alpha_U f = g_U$. This comes from, the exactness of the $\mathcal{G}$ sequence combined with second axiom of sheaves.

In particular, one can find an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X).$$

Consider now the short exact sequence

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0.$$

In particular, you have the previous conditions and $\forall U \subset X$ open set in X, $\forall h \in \mathcal{H}(U)$, $\forall x \in U$, there is $x \in V_x \subset U$ an open set and $\exists g_x \in \mathcal{G}(V_x)$ s.t. $\rho_U^{V_x} h = \beta_{V_x}(g_x)$ (we can find the pre-image in a smaller open set).

Example 3.7 (Short exact sequences)**.**

$$0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} Z^1 \to 0.$$

$$0 \to Z^1 \to \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \to 0.$$

$$0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \to 0.$$

$$0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{1,1} \to 0.$$

$$0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d=\partial} \Omega \to 0.$$

$$0 \to \mathbb{Z} \to \mathcal{E} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{E}^* \to 0.$$

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^* \to 0.$$

Obstructions to lift a global section in $\mathcal{O}^*$ to $\mathcal{O}$. For $f = \exp(2\pi i g_1)$ on U_1 and $f = \exp(2\pi i g_2)$ on U_2, then we can show that $g_1 - g_2 \in \mathbb{Z}$. The obstruction is the first cohomology group of $\mathbb{Z}$. We define cohomology $H^1(\mathcal{F})$ in order to solve a lifting problem of local sections.

Proposition 3.3. *Given a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

for any $U \subset X$ open set we have exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U),$$

but

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0,$$

is not exact.

Let us focus on the particular example

$$0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} Z^1 \to 0.$$

Consider $\omega \in Z^1(X)$, we want to understand the obstruction to find $\varphi \in \mathcal{E}(X)$ s.t. $\omega = d\varphi$ on X.

By Poincaré lemma, for any $x \in X$, there is $V_x \ni x$ an open set in X such that there is $\varphi_x \in \mathcal{E}(V_x)$ such that $\omega|_{V_x} = d\varphi_x$. Thus, there is $(U_i)_{i\in I}$ an open cover of X s.t. there is $\varphi_i \in \mathcal{E}(U_i)$ s.t. $d\varphi_i = \omega|_{U_i}$. On $U_i \cap U_j$, we have $d(\varphi_j - \varphi_i) = 0$, this implies that there exists $\lambda_{ij} \in \mathbb{C}(U_i \cap U_j)$ s.t. $\varphi_j - \varphi_i = \lambda_{ij}$. Let us find a necessary condition on $(\lambda_{ij}) \in \mathbb{C}(U_i \cap U_j)$ which insures that ω admits a global primitive.

Assume there is $\varphi \in \mathcal{E}(X)$, $\omega = d\varphi$. Then on U_i, $\omega = d(\varphi|_{U_i}) = d\varphi_i$ implies that $\varphi|_{U_i} - \varphi_i = \lambda_i \in \mathbb{C}(U_i)$. Similarly we have $\varphi|_{U_j} - \varphi_j = \lambda_j \in \mathbb{C}(U_j)$. Then on $U_i \cap U_j$, we have

$$\lambda_{ij} = \varphi_j|_{U_i\cap U_j} - \varphi_i|_{U_i\cap U_j} = \cdots = \lambda_i - \lambda_j.$$

As we defined λ_{ij}, they are not uniquely defined, but if φ_i' is another primitive on U_i and φ_j' is another primitive on U_j, $d(\varphi_i - \varphi_i') = 0$ suggests there is $\lambda_i' \in \mathbb{C}(U_i)$ s.t. $\lambda_i' = \varphi_i - \varphi_i'$ and there is $\lambda_j' \in \mathbb{C}(U_j)$ s.t. $\lambda_j' = \varphi_j - \varphi_j'$, and let $\lambda_{ij}' = \varphi_i'|_{U_i \cap U_j} - \varphi_j'|_{U_i \cap U_j}$.

$\lambda_{ij}' = \lambda_{ij} - (\lambda_i' - \lambda_j')$ shows that (λ_{ij}) and (λ_{ij}') will differ by $\lambda_i' - \lambda_j'$. By construction, on $U_i \cap U_j \cap U_k$, $\lambda_{ij} + \lambda_{jk} + \lambda_{ki} = 0$, we define with respect to $\mathscr{U} = \bigcup\limits_{i \in I} U_i$,

$$H^1(\mathscr{U}, \mathbb{C}) = \{\lambda_{ij} \in \mathbb{C}(U_i \cap U_j) : \lambda_{ij} + \lambda_{jk} + \lambda_{ki} = \mathbb{C}\}/\{\lambda_i - \lambda_j : \lambda_i \in \mathbb{C}(U_i), \lambda_j \in \mathbb{C}(U_j)\}.$$

We defined a map

$$H^1_{dR}(X) = \mathbb{Z}^1(X)/d\mathcal{E}(X) \to H^1(\mathscr{U}, \mathbb{C}).$$

This map is injective: $\lambda_{ij} = \lambda_i - \lambda_j$, then the local primitives $\varphi_i + \lambda_i \in \mathcal{E}(U_i)$ and $\varphi_j + \lambda_j \in \mathcal{E}(U_i)$ agree on $U_i \cap U_j$, hence they glue in a global smooth function φ s.t. $\omega = \varphi$, hence $\omega \equiv 0$ in $Z^1(X)/d\mathcal{E}(X) = H^1_{dR}(X)$.

In order to prove the surjectivity: consider $\lambda_{ij} \in \mathbb{C}(U_i \cap U_j)$, and using a partition of unity, construct functions $f_i \in \mathcal{E}(U_i)$ and $f_j \in \mathcal{E}(U_j)$ s.t. $\lambda_{ij} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}$ (We used the fact that $H^1(\mathscr{U}, \mathcal{E}) = 0$). Then the local functions f_i and f_j are s.t. df_i and df_j agree on $U_i \cap U_j$, meaning that $df_i = \omega_i \in Z^1(U_i)$ define a global section $\omega \in Z^1(X)$ s.t. $\omega|_{U_i} = \omega_i$ and the cocycle associated to ω w.r.t. $f_i \in \mathcal{E}(U_i)$ is λ_{ij}.

Thus we've shown that the de Rham cohomology is topological invariant and $H^1(\mathscr{U}, \mathcal{E})$ does not depend on the choice of the open cover.

Exercise 3.1. $H^1(\mathscr{U}, \mathcal{E}) = 0$.

Proof. Set $\mathscr{U} = \{U_i\}_{i \in I}$ and for an 1-cocycle $(\lambda_{ij}) \in \mathcal{E}(U_i \cap U_j)$, we define $\lambda_i = \sum\limits_{k \in I} \rho_k \lambda_{ik}$, where ρ_k is the partition of unity subordinated to $\mathscr{U}$. Note that $\rho_k \lambda_{ik}$ is a smooth function, hence in $\mathcal{E}(U_i)$.

$$\lambda_i - \lambda_j = \sum_{k \in I} \rho_k (\lambda_{ik} - \lambda_{jk}) = \sum_{k \in I} \rho_k \lambda_{ij} = \lambda_{ij}.$$

We can also prove that $H^1(X, \mathcal{E}^{0,1}) = 0$ and $H^1(X, \mathcal{C}) = 0$. □

Definition 3.7. *Given a short exact sequence*

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0.$$

Which is the condition $h \in \mathcal{H}(X)$ to admit a pre-image, meaning $\exists g \in \mathcal{G}(X)$ s.t. $\beta_X(g) = h$?

This problem will be a gluing problem with sections in $\mathcal{F}$ and obstruction will be seen as an element of a group $H^1(X, \mathcal{F})$. Consider $X = \bigcup\limits_{i \in I} U_i$ an open cover such that $\forall i$, $\exists g_i \in \mathcal{G}(U_i)$ s.t. $\beta_{U_i}(g_i) = \rho_X^{U_i} h$. On $U_i \cap U_j$, $\beta_{U_i \cap U_j}(g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j}) = 0$, then there is $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ s.t. $\alpha_{U_i \cap U} f_{ij} = g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j}$.

We associated to g an element $f_{ij} \in \prod_{i,j} \mathcal{F}(U_i \cap U_j)$ s.t. $f_{ij} + f_{jk} + f_{ki} \equiv 0$ on $U_i \cap U_j \cap U_k$, with $f_{ij} = -f_{ji}, f_{ii} = 0$ (they are called 1-cocycles*). Cobords are those* *elements* f_{ij} *s.t.* $f_{ij} = f_i|_{U_i\cap U_j} - f_j|_{U_i\cap U_j}$*, where* $f_i \in \mathcal{F}(U_i)$ *and* $f_j \in \mathcal{F}(U_j)$*. Now we define*

$$H^1(\mathscr{U}, \mathcal{F}) = 1\text{-}cocycles/cobords.$$

We proved that there is an injective map from

$$\mathcal{H}(X)/\beta_X(\mathcal{G}(X)) \to H^1(\mathscr{U}, \mathcal{F}).$$

Moreover, one can get rid of $\mathscr{U}$*, by taking other cover* $\mathscr{V}$*: we will say that* $\mathscr{V}$ *is "thiner" than* $\mathscr{U}$ *if for any* $U \in \mathscr{U}$*, there is* $V \in \mathscr{V}$ *s.t.* $V \subset U$*.*

Then we define $H^1(X, \mathcal{F}) = \varinjlim H^1(\mathscr{U}, \mathcal{F})$ *meaning that* $(f_{ij})_{ij} \in \mathcal{F}(U_i \cap U_j)$ *will be* 0 *in* $H^1(X, \mathcal{F})$ *if there is* $\mathscr{U}$ *an open cover s.t.* $H^1(\mathscr{U}, \mathcal{F}) = 0$*. This will imply that* $H^1(\mathscr{V}, \mathcal{F}) = 0$ *for any open cover* $\mathscr{V}$ *thiner than* $\mathscr{U}$*.*

Theorem 3.1. *For any short exact sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *of sheaves, there exists a long exact sequence*

$$\begin{aligned} 0 \to \mathcal{F}(X) \to &\mathcal{G}(X) \to \mathcal{H}(X) \\ &\to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}), \end{aligned}$$

where $\delta : \mathcal{H}(X) \to H^1(X, \mathcal{F})$ *was constructed previously (called* ***cobord operator****) and* $H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$ *is* $f_{ij} \in \mathcal{F}(U_i \cap U_j) \mapsto \alpha(f_{ij}) \in \mathcal{G}(U_i \cap U_j)$*, and the same for* $H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H})$*.*

Proposition 3.4 (**Naturality**)**.** *If there is a commutative diagram for short exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} & \xrightarrow{\beta} & \mathcal{H} & \longrightarrow & 0 \\ & & \big\downarrow F & & \big\downarrow G & & \big\downarrow H & & \\ 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{\alpha'} & \mathcal{G}' & \xrightarrow{\beta'} & \mathcal{H}' & \longrightarrow & 0 \end{array},$$

then we get a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{F}(X) & \xrightarrow{\alpha(X)} & \mathcal{G}(X) & \xrightarrow{\beta(X)} & \mathcal{H}(X) & \xrightarrow{\delta} & H^1(X,\mathcal{F}) & \xrightarrow{\alpha} & H^1(X,\mathcal{G}) & \xrightarrow{\beta} & H^1(X,\mathcal{H}) \\ & & \big\downarrow F(X) & & \big\downarrow G(X) & & \big\downarrow H(X) & & \big\downarrow F_* & & \big\downarrow G_* & & \big\downarrow H_* \\ 0 & \longrightarrow & \mathcal{F}'(X) & \xrightarrow{\alpha'(X)} & \mathcal{G}'(X) & \xrightarrow{\beta'(X)} & \mathcal{H}'(X) & \xrightarrow{\delta} & H^1(X,\mathcal{F}') & \xrightarrow{\alpha} & H^1(X,\mathcal{G}') & \xrightarrow{\beta} & H^1(X,\mathcal{H}') \end{array}.$$

Example 3.8.

$$0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} Z^1 \to 0,$$

gives exact sequence

$$0 \to \mathbb{C}(X) \cong \mathbb{C} \to \mathcal{E}(X) \xrightarrow{d} Z^1(X) \xrightarrow{\delta} H^1(X, \mathbb{C}) \to H^1(X, \mathcal{E}) = 0.$$

Then $Z^1(X)/d\mathcal{E}(X) \cong H^1(X, \mathbb{C})$*.*

Corollary 3.1. $\pi_1(X) = 0$ *hence* $H^1(X, \mathbb{C}) = 0$*.*

Example 3.9.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{E} & \xrightarrow{\exp(2\pi i\cdot)} & \mathcal{E}^* & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\subset & & \downarrow\subset & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C} & \xrightarrow{\exp(2\pi i\cdot)} & \mathcal{C}^* & \longrightarrow & 0 \end{array},$$

gives a long exact sequence

$$\begin{array}{ccccccc} \mathcal{E}(X) & \xrightarrow{\exp(2\pi i\cdot)} & \mathcal{E}^*(X) & \xrightarrow{\delta} & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{E}) = 0 \\ \downarrow\subset & & \downarrow\subset & & \downarrow\cong & & \\ \mathcal{C}(X) & \xrightarrow{\exp(2\pi i\cdot)} & \mathcal{C}^*(X) & \xrightarrow{\delta} & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{C}) = 0 \end{array},$$

shows that

$$H^1(X,\mathbb{Z}) = \mathcal{C}^*(X)/\exp(2\pi i\mathcal{C}(X)) = \mathcal{E}^*(X)/\exp(2\pi i\mathcal{E}(X)).$$

Corollary 3.2. $\pi_1(X) = 0$ *hence* $H^1(X,\mathbb{Z}) = 0$.

Example 3.10. ***Dolbeault isomorphism.***

$$0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \to 0,$$

gives exact sequence

$$\mathcal{E}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1}(X) \xrightarrow{\delta} H^1(X,\mathcal{O}) \to H^1(X,\mathcal{E}) = 0.$$

Then we get the Dolbeault isomorphism $\mathcal{E}^{0,1}(X)/\overline{\partial}\mathcal{E}(X) \cong H^1(X,\mathcal{O})$.

$H^0(X,\mathcal{O}) \cong \operatorname{Ker}\overline{\partial} : \mathcal{E} \to \mathcal{E}^{0,1}$, $H^1(X,\mathcal{O}) \cong \operatorname{Coker}\overline{\partial}$.

We will prove Riemann-Roch theorem which shows the difference of the dimensions of $H^0(X,L)$ and $H^1(X,L)$ is a topological invariant (first result of index theory).

We will prove that $\dim H^1(X,\mathcal{O})$ is finite and $\dim H^1(X,\mathcal{O}) = g$ is topological invariant (while $\dim H^1(X,\mathbb{C}) = 2g$).

Corollary 3.3. *Any Riemann surface of genus g admits a nonconstant meromorphic function with a unique pole of order at most $g+1$.*

Proof.

$$0 \to \mathcal{O} \to \mathcal{M} \xrightarrow{p} \mathcal{M}/\mathcal{O} \to 0,$$

where p is a projection and $\mathcal{M}/\mathcal{O}$ is the sheaf of "polar parts". Sections of $\mathcal{M}/\mathcal{O}$ are local meromorphic functions and we decide $f \in \mathcal{M}(U)$ and $g \in \mathcal{M}(V)$ define the same section of $(\mathcal{M}/\mathcal{O})(U\cap V)$ if there is $t \in \mathcal{O}(U\cap V)$ s.t. $(f-g)|_{U\cap V} = t$.

$(\mathcal{M}/\mathcal{O})(X)$ are polar parts: such a global section it is a data given by $P_1,\cdots,P_n \in X$ and prescribed polar parts: in each P_i we consider in local coordinate z_i s.t. $z_i(P_i) = 0$,

$$\frac{a_d}{z_i^d} + \frac{a_{d-1}}{z_i^{d-1}} + \cdots + \frac{a_1}{z_i^1},$$

it is a polynomial of degree d in $\frac{1}{z_i}$ with no constant term. Note that $\dim(\mathcal{M}/\mathcal{O})(X) = \infty$.

The short exact sequence gives

$$\mathcal{M}(X) \xrightarrow{p_X} (\mathcal{M}/\mathcal{O})(X) \xrightarrow{\delta_X} H^1(X, \mathcal{O}).$$

Since $\dim H^1(X, \mathcal{O})$ is finite, then $\dim \operatorname{Ker} \delta$ is infinite hence $\dim \operatorname{Im} p_X$ is infinite.

Let us consider a given point $P \in X$ and the vector space in $(\mathcal{M}/\mathcal{O})(X)$ generated by $\{\frac{1}{z^{g+1}}, \cdots \frac{1}{z}\}$, where z is a local coordinate at $P \in X$ s.t. $z(P) = 0$. This implies there are $\lambda_{g+1}, \cdots, \lambda_1 \in \mathbb{C}$ s.t. $\sum\limits_{i=1}^{g+1} \frac{\lambda_i}{z^i} \in \operatorname{Ker} \delta = \operatorname{Im} p_X$. Thus there is $f \in \mathcal{M}(X)$, with a unique pole at P and polar parts at P being $\sum\limits_{i=1}^{g+1} \frac{\lambda_i}{z^i}$. $\square$

Corollary 3.4. *If X is a compact Riemann surface of genus $g = 0$, then X is biholomorphic to $\mathbb{P}^1$.*

Proof. $f \in \mathcal{M}(X)$ non constant and have a unique pole $P \in X$ of order 1 (at most 1 and non constant). Such a map is a holomorphic map $f : X \to \mathbb{P}^1$ of degree 1 (since ∞ has a unique pre-image), hence f is an isomorphism (f' does not vanish). $\square$

Corollary 3.5. *If $g = 1$, there exists a meromorphic $f \in \mathcal{M}(X)$ with a unique pole 2 (in fact ≤ 2 but the order being 1 is the last case).*

Weierstrass $\wp$-function (exercise).

Remark 3.4. *If we have a non-constant holomorphic map $f : X \to \mathbb{P}^1$, it is surjective due to closeness and openness of f.*

If we admit $\dim H^1(X, \mathcal{L})$ is finite then we have the following general theorem.

Theorem 3.2. *Any holomorphic line bundle $\mathcal{L}$ over a Riemann surface admits infinitely many meromorphic sections. More precisely, for any $P \in X$, there exists a meromorphic section s of $\mathcal{L}$ such that s admits a unique pole at $P \in X$ with order at least 1 and at most $\dim H^1(X, \mathcal{L}) + 1$.*

Proof.

$$0 \to \mathcal{O}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{L}} \xrightarrow{p} \mathcal{M}_{\mathcal{L}}/\mathcal{O}_{\mathcal{L}} \to 0,$$

where $\mathcal{O}_{\mathcal{L}}$ is the sheaf of local holomorphic sections of $\mathcal{L}$ and $\mathcal{M}_{\mathcal{L}}$ is the sheaf of local meromorphic sections of $\mathcal{L}$. The fibers at $x \in X$ of $\mathcal{M}_{\mathcal{L}}/\mathcal{O}_{\mathcal{L}}$ is the quotient of

$$\{(U, s) : U \text{ open neighborhood of } x \text{ in } X, s \in \mathcal{M}_{\mathcal{L}}(U)\}/\sim,$$

where $(U, s) \sim (V, t)$ when $t - s$ is a holomorphic section on $W \subset U \cap V$.

$(\mathcal{M}_{\mathcal{L}}/\mathcal{O}_{\mathcal{L}})(X)$ is the space of polar parts of sections of $\mathcal{L}$. Such a polar part at $P \in X$ is given in a local coordinate z such that $z(P) = 0$ and with respect to a local holomorphic section t of $\mathcal{L}$ by the following data

$$(\sum_{i=1}^{d} \frac{a_i}{z^i})t, a_i \in \mathbb{C}.$$

We choose $d = \dim H^1(X, \mathcal{L}) + 1$, then the exact sequence

$$\mathcal{M}_{\mathcal{L}}(X) \xrightarrow{p_X} (\mathcal{M}_{\mathcal{L}}/\mathcal{O}_{\mathcal{L}})(X) \xrightarrow{\delta_X} H^1(X, \mathcal{O}_{\mathcal{L}}),$$

shows that there exists $a_i \in \mathbb{C}$ such that the corresponding polar part is in $\operatorname{Ker} \delta_X = \operatorname{Im} p_X$. $\square$

An alternative proof using the Dolbeault operator. Recall $\overline{\partial}_{\mathcal{L}} : \mathcal{C}^\infty(U, \mathcal{L}) \to \mathcal{C}^\infty(U, \mathcal{L} \otimes \mathcal{E}^{0,1})$. For any $U \subset X$, locally we can define $\overline{\partial}_{\mathcal{L}}(ft)$ as being $\overline{\partial}(f)t$, where f is a local smooth section and t a holomorphic trivial of $\mathcal{L}$. If s is another holomorphic trivial and $s = ht$ where h is a holomorphic function, then $\overline{\partial}_{\mathcal{L}}(fs) = \overline{\partial}_{\mathcal{L}}(fht) = ht\overline{\partial}(f) = \overline{\partial}(f)s$. Thus $\overline{\partial}_{\mathcal{L}}$ is well-defined.

$$0 \to \mathcal{O}_{\mathcal{L}} \to \mathcal{E}_{\mathcal{L}} \xrightarrow{\overline{\partial}_{\mathcal{L}}} \mathcal{L} \otimes \mathcal{E}^{0,1} \to 0$$

gives a long exact sequence

$$\mathcal{E}_{\mathcal{L}}(X) \xrightarrow{\overline{\partial}_{\mathcal{L}}} (\mathcal{E}^{0,1} \otimes \mathcal{L})(X) \xrightarrow{\delta} H^1(X, \mathcal{O}_{\mathcal{L}})(\cong H^1(X, \mathcal{L})).$$

Then we have the Dolbeault isomorphism

$$(\mathcal{E}^{0,1} \otimes \mathcal{L})(X)/\overline{\partial}_L \mathcal{E}_{\mathcal{L}}(X) \cong H^1(X, \mathcal{L}).$$

Denote by $d = \dim H^1(X, \mathcal{O}_{\mathcal{L}}) + 1$ and choose a point $P \in X$ with a local coordinate z such that $z(P) = 0$, and t a local holomorphic trivialization of $\mathcal{L}$ at P.

Consider for any $i \in \{1, \cdots, d\}$, the local section $\rho \frac{t}{z^i}$ with ρ a bump function. Then $\rho \frac{t}{z^i} \in \mathcal{E}_{\mathcal{L}}(X \setminus \{P\})$ and $\overline{\partial}_{\mathcal{L}}(\rho \frac{t}{z^i} t) = \overline{\partial}(\rho) \frac{t}{z^i} t$ on $X \setminus \{P\}$ but $\overline{\partial}(\rho) = 0$ in the neighborhood of P, hence $\overline{\partial}(\rho \frac{t}{z^i} t) \in (\mathcal{L} \otimes \mathcal{E}^{0,1})(X)$.

Thus there are $\lambda_1, \cdots, \lambda_d \in \mathbb{C}$ such that $\sum\limits_{i=1}^{d} \lambda_i (\overline{\partial}\rho) \frac{t}{z^i} \equiv 0$ in $(\mathcal{E}^{0,1} \otimes \mathcal{L})(X)/\overline{\partial}_L \mathcal{E}_{\mathcal{L}}(X)$. Thus there is $\beta \in \mathcal{E}_{\mathcal{L}}(X)$ s.t. $\overline{\partial}_{\mathcal{L}} \beta = \sum\limits_{i=1}^{d} \lambda_i (\overline{\partial}\rho) \frac{t}{z^i}$, hence denote $s = \sum\limits_{i=1}^{d} \lambda_i \rho \frac{t}{z^i}$ we have $s - \beta$ is holomorphic. Thus s is meromorphic section of $\mathcal{L}$ with prescribed polar part. $\square$

3.3 Line bundles and divisors

X a compact Riemann surface.

Definition 3.8. *A **divisor** on X is an element of the free abelian group* $\operatorname{Div}(X)$ *generated by points in X. Such an element is given by*

$$D = \sum_{i=1}^{k} n_i \cdot P_i, \quad P_i \in X, n_i \in \mathbb{Z}.$$

We define $\deg D = \sum\limits_{i=1}^{k} n_i$. *If for any $1 \le i \le k$, we have $n_i \in \mathbb{N}^*$, we say D is an effective divisor. $m_{P_i}(D) = n_i \in \mathbb{Z}$ and $m_Q(D) = 0$ if $Q \notin \{P_1, \cdots, P_k\}$.*

We say $D_1 \ge D_2$ iff $D_1 - D_2 \ge 0$, that is $m_P(D_1 - D_2) \ge 0$ for any $P \in X$ ($D_1 - D_2$ is an effective divisor).

For any $f \in \mathcal{M}(X)$, we associated the divisor

$$\operatorname{div}(f) = \sum_{P \in X} v_P(f) \cdot P, \quad v_P(f) = \begin{cases} = 0, & P \textit{ is not a zero or a pole}; \\ = m_i \in \mathbb{N}^*, & P \textit{ is a zero of order } m_i; \\ = -n_i \in \mathbb{Z} \setminus \mathbb{N}, & P \textit{ is a pole of order } n_i. \end{cases}$$

For $f, g \in \mathcal{M}(X)$, we have

$$v_P(fg) = v_P(f) + v_P(g) \Rightarrow \operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g).$$

$$v_P(f^{-1}) = -v_P(f) \Rightarrow \operatorname{div}(f^{-1}) = -\operatorname{div}(f).$$

$\operatorname{div}(f)$ *is effective iff f is holomorphic.*

Definition 3.9. *We do the same with $(\mathcal{L}, f)$, where $\mathcal{L}$ is a holomorphic line bundle over X and $f \in \mathcal{M}_{\mathcal{L}}(X)$, $\operatorname{div}(f) = \sum_{P \in X} v_P(f)$ is well-defined divisor (this does not depend upon the local trivialization).*

We have $f^{-1} \in \mathcal{M}_{\mathcal{L}^*}(X)$ and $f \in \mathcal{M}_{\mathcal{L}_1}(X)$ and $g \in \mathcal{M}_{\mathcal{L}_2}(X)$ give that $f \cdot g \in \mathcal{M}_{\mathcal{L}_1 \otimes \mathcal{L}_2}(X)$.

We will say that $(\mathcal{L}_1, s)$ and $(\mathcal{L}_2, t)$ are equivalent w.r.t. $\mathcal{L}_1, \mathcal{L}_2$ holomorphic line bundles, $s \in \mathcal{M}_{\mathcal{L}_1}(X)$ and $t \in \mathcal{M}_{\mathcal{L}_2}(X)$, if there exists an isomorphism

$$\varphi : \mathcal{L}_1 \to \mathcal{L}_2, \quad \textit{such that } \varphi(s) = t.$$

Proposition 3.5. *Two pairs $(\mathcal{L}_1, s)$ and $(\mathcal{L}_2, t)$ which are equivalent, define the same divisor* $\operatorname{div}(s) = \operatorname{div}(t)$ *and the map*

$$(\mathcal{L}_1, s)/\sim \to \operatorname{Div}(X),$$

is a bijection which is a group homomorphism.

Proof. $(\mathcal{L}_1, s) \sim (\mathcal{L}_2, t)$ iff $t \cdot s^{-1}$ is a holomorphic section of $L_1^* \otimes L_2$ which does not vanish. Then $\operatorname{div}(t \cdot s^{-1}) = 0$ hence $\operatorname{div}(t) = \operatorname{div}(s)$.

The above shows that the map is injective. Let us prove now the surjectivity. Take $D = \sum_{i=1}^{k} n_i \cdot P_i$, consider $\mathscr{U} = (U_0, \cdots, U_k)$ such that U_i and U_j are disjoint for $i, j > 0$, and $U_0 = X \setminus \{P_1, \cdots, P_k\}$. $U_i = D_i$ is a small disk centered at P_i with local coordinate z_i with $z_i(P) = 0$.

Consider the transition functions $\varphi_i = \varphi_{U_i U_0} = z_i^{n_i}$ holomorphic on $z_i \neq 0$. There is no cocycle condition since $U_i \cap U_j \cap U_l = \varnothing$. The cocycle defines a holomorphic line bundle and by construction the section $\equiv 1$ on U_0, is meromorphic section of the line bundle with zeros and poles at P_i s.t. the associated divisor is $D = \sum_{i=1}^{k} n_i \cdot P_i$.

□

Definition 3.10. *The pre-image of the divisor $D \in$* $\operatorname{Div}(X)$ *is a given holomorphic line bundle called $\mathcal{O}_D$ which comes with a meromorphic section 1_D such that* $\operatorname{div}(1_D) = D$.

For example $\mathcal{O}$ is associated to $D = 0$.

Moreover if two divisor D and D' define the same holomorphic line bundle $\mathcal{O}_D$ this implies that there are two meromorphic sections s_1 and s_2 of $\mathcal{O}_D$ such that $\mathrm{div}(s_1) = D$ and $\mathrm{div}(s_2) = D'$. But there is $f \in \mathcal{M}(X)$ s.t. $fs_1 = s_2$ and $\mathrm{div}(fs_1) = \mathrm{div}(f) + \mathrm{div}(s_1)$ hence $D' = \mathrm{div}(f) + D$. Thus, there is a bijection map

$$\text{Holomorphic line bundles} \to \text{Divisors}/\sim .$$

$\sim$ is called linear equivalent, $D \equiv D'$ iff there is $f \in \mathcal{M}(X)$ s.t. $D = D' + \mathrm{div}(f) = D' + (f)$.

Denote by $\deg(D) = \sum_{i=1}^{k} n_i$, then $D = D' + (f)$ gives that $\deg(D) = \deg(D')$ because $\deg(f) = 0$.

Exercise 3.2. *A meromorphic function on a compact Riemann surface admits as many zeros as poles.*

Proof. A non-constant meromorphic function f can be seen as $f : X \to \mathbb{P}^1$ and it is surjective. We want to show the number of $f^{-1}(0)$ equals the number of $f^{-1}(\infty)$ (WAIT).

Another proof is given by considering $\frac{f'}{f}$, please check Proposition 3.8.

A more general result is Corollary 3.7. $\square$

Mittag-Leffler Problem

$$H^0(X, \mathcal{O}_D) = \{f \in \mathcal{M}(X) : f \cdot z_i^{n_i} \text{ is holomorphic}\}.$$

Then $v_{P_i}(f) \geq -n_i$, i.e. $v_{P_i}(f) + n_i \geq 0$,

$$H^0(X, \mathcal{O}_D) = \{f \in \mathcal{M}(X) : \mathrm{div}(f) \geq -D\}.$$

$\mathcal{O}_D$ is trivial iff there is $s \in H^0(X, \mathcal{O}_D)$ non-vanishing, that is $s \in \mathcal{M}(X)$ with $\mathrm{div}(s) = -D$, i.e. $\mathrm{div}(s^{-1}) = D$.

Theorem 3.3 (Mittag-Leffler Problem)**.** *Let us consider $P_1, \cdots, P_k, Q_1, \cdots, Q_l \in X$ and $m_i \in \mathbb{N}$, $n_j \in \mathbb{N}$, we are trying to find $f \in \mathcal{M}(X)$ such that f admits poles at P_i of order at most m_i and zeros at Q_j of order at least n_j.*

That is $v_{P_i}(f) \geq -m_i$ and $v_{Q_j}(f) \geq n_j$, that is $\mathrm{div}(f) \geq -D$, where $D = \sum_{i,j} m_i P_i - n_j Q_j$. That is $f \in H^0(X, \mathcal{O}_D)$.

We have seen in Theorem 3.2 that for $g = 0$, there is $s \in H^0(X, \mathcal{O}_P) \setminus H^0(X, \mathcal{O})$ for any $P \in X$ and $D = -P$.

Notice that the divisor associated to a holomorphic section is effective.

Proposition 3.6. *The degree of a line bundle is a topological invariant.*

We have defined $\deg(D) = \sum n_i$, where $D = \sum n_i P_i$ and we define $\deg(\mathcal{O}_D) = \deg(D)$ (this does not depend on the section because if $\mathcal{L} = \mathcal{O}_D$ and $\mathcal{L} = \mathcal{O}_{D'}$ we have seen that there is $f \in \mathcal{M}(X)$ s.t. $D = D' + (f)$, hence $\deg(D) = \deg(D')$ since $\deg(f) = 0$).

Topological definition of degree of complex line bundle

The degree can be defined for any topological complex line bundle over a Riemann surface. Take $\mathcal{L}$ a complex line bundle and consider a continuous section s with a finite number of isolated zeros $P_1, \cdots, P_k \in X$ (s will trivialize the complex bundle on $X \setminus \{P_1, \cdots, P_k\}$).

In the neighborhood of each $P_i \in X$, consider the local coordinate z_i s.t. $z_i(P_i) = 0$ and the bundle is trivialized over $\{|z_i| < 1\}$. For $\varepsilon > 0$ small enough, s does not vanish on $|z_i| = \varepsilon$ and we consider the map

$$s|_{|z_i|=\varepsilon} : |z_i| = \varepsilon \cong \mathbb{S}^1 \xrightarrow{\tilde{s}_{P_i}} \mathbb{C}^* \cong \mathbb{S}^1$$

and define $\mathrm{ind}_{P_i}(s) = \deg(\tilde{s}_{P_i})$. We define

$$\deg(\mathcal{L}) = \deg(s) = \sum_{i=1}^{k} \mathrm{ind}_{P_i}(s),$$

it is a topological invariant.

Exercise 3.3. $\deg(\mathcal{L})$ *does not depend on the section s.*

Proof. If s_0 and s_1 are two sections of $\mathcal{L}$, consider $s_t = (1-t)s_0 + ts_1$, $t \in [0,1]$. If we can show that $\deg(s_t)$ is continuous on t, then since it's in a discrete group $\mathbb{Z}$, we have $\deg(s_0) = \deg(s_1)$. □

Exercise 3.4. $\deg(\mathcal{O}_D) = \deg(D)$. *Indication: consider the meromorphic section 1_D and associate the coordinates section $s = \frac{1_D}{1+\|1_D\|^2}$, $\deg(s) = D$.*

Exercise 3.5. $\deg(\mathcal{L}) = 0$ *iff $\mathcal{L}$ is topologicall trivial.*

Proposition 3.7. *Let $\mathcal{L}$ be a holomorphic line bundle with* $\deg(\mathcal{L}) < 0$*, then $\mathcal{L}$ does not admit non trivial holomorphic sections.*

Proof. If $s \in H^0(X, \mathcal{L})$ is a holomorphic section, $\deg(\mathcal{L}) = \deg(s) \geq 0$. □

Proposition 3.8. *Let X be a compact Riemann surface and $\omega \in \Omega^1_{mer}(X)$ a meromorphic 1-form on X (ω is a meromorphic section of T^*X). Then $\sum\limits_{P \in X} \mathrm{Res}_P\, \omega = 0$.*

Definition 3.11. *$T^*X = KX$ is called the* ***canonical bundle*** *and* $\mathrm{div}(\omega)$ *is usually called a* ***canonical divisor****: its class is denoted by K, meaning $T^*X = \mathcal{O}_K$.*

Corollary 3.6. *For any $f \in \mathcal{M}(X)$, $\frac{df}{f} \in \Omega^1_{mer}(X)$ and*

$$\mathrm{Res}_P(\frac{df}{f}) = \begin{cases} 0, & P \textit{ is not a zero nor a pole;} \\ n \in \mathbb{N}^*, & P \textit{ is a zero of order } n; \\ -m \in \mathbb{Z} \setminus \mathbb{N}, & P \textit{ is a pole of order } m. \end{cases}$$

Then $\mathrm{div}(f)$ *is of degree zero.*

Proof of the proposition. $\operatorname{Res}_P(\omega)$ is well-defined since if P is a pole of ω, consider z a local coordinate at P s.t. $z(P) = 0$ and consider

$$\omega = [\frac{a_d}{z^d} + \frac{a_{d-1}}{z^{d-1}} + \cdots + \frac{a_1}{z} + f(z)]dz,$$

with $a_i \in \mathbb{C}$ and $f \in \mathcal{O}(U)$. The point is that a_1 does not depend on the local coordinate z and is $\frac{1}{2\pi i}\int_{|z_i|=\varepsilon}\omega$, for ε small enough, and by definition $a_1 = \operatorname{Res}_P \omega$.

$$\sum_{P\in X} \operatorname{Res}_P \omega = \sum_{i=1}^{k} \int_{|z_i|=\varepsilon} \omega =_{\text{Stokes}} \int_{X\setminus \bigcup\limits_{i=1}^{k}\{|z_i|\leq\varepsilon\}} d\omega = 0.$$

□

Corollary 3.7. *For $f \in \mathcal{M}(X)$, $\omega = \frac{df}{f-a}$ gives the number of poles is the number of $f^{-1}(\{a\})$, for any $a \in \mathbb{C}$.*

Hence for any $a \in \mathbb{C}$, $\# f^{-1}(\{a\})$ is the same, now we define it as $\deg(f)$.

3.4 Riemann-Roch theorem

Reminder: all holomorphic line bundles over a Riemann surface X admit meromorphic sections. If $\mathcal{L}$ is a holomorphic line bundle and s is a meromorphic sections, we call $D = \operatorname{div}(s)$ and $\mathcal{L} \cong \mathcal{O}_D$. If $s' = fs$ is another meromorphic section of $\mathcal{L}$ with $f \in \mathcal{M}(X)$ (because $s' \cdot s^{-1}$ is a section of $\mathcal{L} \otimes \mathcal{L}^* = X \times \mathbb{C}$ trivial). $\operatorname{div}(s') = \operatorname{div}(f) + \operatorname{div}(s)$, hence $D' := \operatorname{div}(s') = \operatorname{div}(s) + (f) = D + (f)$, which we say $D' \sim D$ are linear equivalent and (f) is a **principal divisor**.

$\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2} = \mathcal{O}_{D_1+D_2}$ because if 1_{D_1} is a meromorphic section of $\mathcal{O}_{D_1}$ such that $\operatorname{div}(1_{D_1}) = D_1$ and 1_{D_2} is a meromorphic section of $\mathcal{O}_{D_2}$ such that $\operatorname{div}(1_{D_2}) = D_2$, then $1_{D_1} \cdot 1_{D_2}$ is a meromorphic section of $\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}$ and $\operatorname{div}(1_{D_1} \cdot 1_{D_2}) = \operatorname{div}(1_{D_1}) + \operatorname{div}(1_{D_2}) = D_1 + D_2$.

One classical notation: $\mathcal{O}_K \otimes \mathcal{O}_D = \Omega_D$.

$$\Omega_D(U) = \{\omega \text{ meromorphic sections of } \Omega(U) \text{ such that } \operatorname{div}(\omega) \geq -D\}.$$

$$(\mathcal{O}_K \otimes \mathcal{O}_D)(U) = \mathcal{O}_K(U) \otimes \mathcal{O}_D(U) = \Omega(U) \otimes \mathcal{O}(U).$$

Theorem 3.4 (**Riemann-Roch theorem**)**.** *Let D be any divisor on a compact Riemann surface of genus g, then*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg(D).$$

It is equivalent to say that for any holomorphic line bundle $\mathcal{L}$,

$$\dim H^0(X, \mathcal{O}_{\mathcal{L}}) - \dim H^1(X, \mathcal{O}_{\mathcal{L}}) = 1 - g + \deg(\mathcal{L}).$$

Corollary 3.8 (**Riemann**)**.** $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) \geq 1 - g + \deg(D)$*.*

We shall admit the following theorem in advance.

Theorem 3.5 (**Serre duality**)**.** $H^1(X, \mathcal{O}_D)^* \cong H^0(X, \mathcal{O}_{K-D}) = H^0(X, \Omega_{-D})$*.*

Then $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) = \dim_{\mathbb{C}} H^0(X, \Omega_{-D})$*.*

Some interpretation of Riemann-Roch theorem

For $D = n_1P_1 + \cdots + n_kP_k$ an effective divisor, $n_i \in \mathbb{N}^*$,

$$\Omega_{-D} = \{s \in H^0(X, \Omega) : s \text{ vanishes at each } P_i \text{ of order } \geq n_i\}.$$

Riemann-Roch will state:

$$\dim H^0(X, \mathcal{O}_D) - \dim H^0(X, \Omega_{-D}) = 1 - g + \deg(D).$$

This implies

$$\dim H^0(X, \mathcal{O}_D) = 1 - (g - \dim H^0(X, \Omega_{-D})) + \deg(D).$$

While

$$H^0(X, \mathcal{O}_D) = \{f \in \mathcal{M}(X) : \mathrm{div}(f) \geq -D\},$$

that is, f admits poles at P_i of order at most n_i.

There is a natural map the polar part map

$$\mathcal{M}(X) \supset H^0(X, \mathcal{O}_D) \to \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \oplus \cdots \oplus \mathbb{C}^{n_k} = \mathbb{C}^{\deg(D)}.$$

The map at P_i is given by the polar part

$$f \mapsto \frac{a_{i,n_i}}{z_i^{n_i}} + \cdots + \frac{a_{i,1}}{z_i},$$

where z_i is a local coordinate at P_i with $z_i(P_i) = 0$ and $a_{i,l} \in \mathbb{C}$.

We want to understand the image of this map. Also notice that the polar part map is well-defined and injective on $H^0(X, \mathcal{O}_D)/\mathcal{O}(X)$, meaning that two meromorphic functions which have the same polar part differ by a holomorphic function on X, hence by a constant.

Recall that the residue theorem, for any $f \in \mathcal{M}(X)$, any $\omega \in \Omega(X)$ (holomorphic 1-form), we have $\sum_{x\in X} \mathrm{Res}_x(f\omega) = 0$. For a polar part there is an obstruction to be lifted to a global defined $f \in \mathcal{M}(X)$ which is, $\forall \omega \in \Omega(X)$ denoted by $\omega(z_i) = \sum_{j=0}^{\infty} b_{ij} z_i^j dz_i$ the local expansion in power series at each P_i. Since $\sum_{x\in X} \mathrm{Res}_x(f\omega) = 0$, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{i,j} \cdot b_{i,-1-j} = 0.$$

Example 3.11. *$D = P$ (hence $k = 1$ and $n_1 = 1$), then the condition on the polar part at $P : \frac{\lambda}{z}$ is $\mathrm{Res}\left(\frac{\lambda}{z}\omega\right) = 0$, i.e. $\lambda \cdot b_0 = 0$, where b_0 is such that $\omega(z) = (b_0 + b_1 z + \cdots)dz$.*

Moreover this implies that all forms $\omega \in \Omega_{-D}(X)$ give trivial condition because they vanish at order n_i at each P_i, i.e. $b_{i,j} = 0, \forall j \leq n_i - 1$.

The image of polar part map is the space realizing the non trivial conditions. Therefore, the image of the polar part map is of dimension

$$\deg(D) - [\dim_{\mathbb{C}} H^0(X, \Omega) - \dim H^0(X, \Omega_{-D})] = \deg(D) - g + \dim H^1(X, \mathcal{O}_D).$$

Here

$$\dim H^0(X,\Omega) = \dim H^0(X,\mathcal{O}_K) = \dim H^1(X,\mathcal{O}_{K-K}) = \dim H^1(X,\mathcal{O}) = 0.$$

Since the polar part map is injective on $H^0(X,\mathcal{O}_D)/\mathbb{C}$, hence the image is of dimension $\dim_{\mathbb{C}} H^0(X,\mathcal{O}_D) - 1$. Thus

$$\dim_{\mathbb{C}} H^0(X,\mathcal{O}_D) - \dim_{\mathbb{C}} H^1(X,\mathcal{O}_D) = 1 - g + \deg(D).$$

Proof of Riemann-Roch theorem

Proof of Riemann-Roch theorem. Notice that for $D = 0$, we have

$$\dim H^0(X,\mathcal{O}) - \dim H^1(X,\mathcal{O}) = 1 - g + 0.$$

We will prove now that Riemann-Roch is true for D if and only if Riemann-Roch is true for $D + P$ for any $P \in X$.

We need to prove that

$$\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) = \dim H^0(X,\mathcal{O}_{D+P}) - \dim H^1(X,\mathcal{O}_{D+P}) - 1.$$

There is a short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathcal{S}_P \to 0,$$

where $\mathcal{S}_P$ is the sky scraper sheaf defined in Example 3.4.

First case, $d \in \mathbb{N}^*$ is the coefficient of P in $D + P$ and then the coefficient of P in D is $d - 1 \in \mathbb{N}$

$f \in \mathcal{O}_{D+P}(U)$ with $P \in U$ and consider at a local coordinate z at P the polar part of f as being $\frac{a_1}{z} + \cdots + \frac{a_d}{z^d}$. Then $f \in \mathcal{O}_D(U)$ if and only if $a_d = 0$. Define $\mathcal{O}_{D+P}(U) \to \mathcal{S}_P(U)$, $f \mapsto a_d$.

By construction we have a short exact sequence. This gives a long exact sequence:

$$0 \to H^0(X,\mathcal{O}_D) \to H^0(X,\mathcal{O}_{D+P}) \to H^0(X,\mathcal{S}_P) \to H^1(X,\mathcal{O}_D) \to H^1(X,\mathcal{O}_{D+P}) \to H^1(X,\mathcal{S}_P) \to 0.$$

This gives the alternate sum of dimensions

$$\begin{aligned} 0 = \dim H^0(X,\mathcal{O}_D) - \dim H^0(X,\mathcal{O}_{D+P}) + \dim H^0(X,\mathcal{S}_P) \\ - \dim H^1(X,\mathcal{O}_D) + \dim H^1(X,\mathcal{O}_{D+P}) - \dim H^1(X,\mathcal{S}_P). \end{aligned}$$

Since $\dim H^0(X,\mathcal{S}_P) = 1$ and $\dim H^1(X,\mathcal{S}_P) = 0$, we have

$$0 = \dim H^0(X,\mathcal{O}_D) - \dim H^0(X,\mathcal{O}_{D+P}) + 1 - \dim H^1(X,\mathcal{O}_D) + \dim H^1(X,\mathcal{O}_{D+P}).$$

Similar proof works for negative coefficient case of D, and works for $D - P$. $\square$

Corollary 3.9. $\deg(K) = 2g - 2$.

Proof. Consider $D = K,\ H^0(X, \mathcal{O}_D) = H^0(X, \mathcal{O}_K) = \Omega(X)$.

$$\dim \Omega(X) = \dim H^0(X, \Omega) = \dim H^1(X, \mathcal{O}) = g.$$

$$\dim H^1(X, \mathcal{O}_K) = \dim H^0(X, \mathcal{O}_{K-K}) = \dim H^0(X, \mathcal{O}) = 1.$$

Then

$$g - 1 = 1 - g + \deg(K).$$

Then $\deg K = 2g - 2$. □

Corollary 3.10. *If* $\deg(D) > 2g - 2$*, then*

$$\dim H^0(X, \mathcal{O}) = 1 - g + \deg(D), \quad \textit{and } \dim H^1(X, \mathcal{O}_D) = 0.$$

Proof. $(H^1(X, \mathcal{O}_D))^* \cong H^0(X, \mathcal{O}_{K-D})$ and $\deg(K - D) = 2g - 2 - \deg(D) < 0$ hence $H^0(X, \mathcal{O}_{K-D}) = 0$. □

Example 3.12. *Another particular case* $\deg(D) = 0$.

Then $H^0(X, \mathcal{O}_D)$ *either admits no nontrivial section or if there is, this section does not vanish. Then we have two cases*

$$\begin{cases} \dim H^0(X, \mathcal{O}_D) = 0 \\ \dim H^1(X, \mathcal{O}_D) = g - 1 \end{cases} \quad \textit{or} \quad \begin{cases} \dim H^0(X, \mathcal{O}_D) = 1 \\ \dim H^1(X, \mathcal{O}_D) = g \end{cases}.$$

Theorem 3.6 (**Riemann-Hurwitz theorem**). *X and Y are Riemann surfaces of genus $g(X)$ and $g(Y)$ respectively. Let $f : X \to Y$ be a holomorphic non-constant map, f is an open map and since X is compact f will be surjective, with finite fiber and f is a ramified cover.*

For any $x \in X$, there is holomorphic chart coordinate in x and a holomorphic chart centered at $f(x)$ such that f reads: $z \to z^d, d \geq 1$. $d \in \mathbb{N}$ does not depend on local coordinates and it is called the ***ramification degree*** *at x and denoted by $l(x)$.*

The set $\mathcal{R} = \{x \in X : l(x) > 1\}$ is called the ***ramification set*** *and $\mathcal{R}$ is a finite set. Moreover $X \setminus f^{-1}(f(\mathcal{R})) \to Y \setminus f(\mathcal{R})$ is a cover. The degree of the cover is called* $\deg(f)$ *(the number of sheets),* $\deg(f) = \sum\limits_{x \in f^{-1}(y)} l(x)$ *for any $y \in Y \setminus f(\mathcal{R})$.*

(i) $\chi(X) = 2 - 2g(X) = \deg(f)(\chi(Y)) - \sum\limits_{i \in X} (l(x) - 1)$.

(ii) $2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum\limits_{x \in X} (l(x) - 1)$.

Example 3.13. $\wp : \mathbb{C}/\Lambda \to \mathbb{P}^1$, $0 = 2 \cdot (0 - 2) + 4 \cdot 1$.

Corollary 3.11. *If $g(Y) > g(X)$, then there is no non-constant holomorphic map $f : X \to Y$.*

Corollary 3.12. $\sum\limits_{x \in X} (l(x) - 1) \in 2\mathbb{Z}$.

Proof of Riemann-Hurwitz theorem. Let us take ω a meromorphic 1-form on Y. Then $\operatorname{div}(\omega) = \sum_{P\in Y} \operatorname{ord}_P(\omega)\cdot P$ is the canonical divisor of degree $2g(Y)-2$.

Now consider $f^*\omega$ as a meromorphic section of K_X,

$$\operatorname{ord}_x(f^*\omega) = l(x)\operatorname{ord}_{f(x)}\omega + (l(x)-1).$$

In order to prove this formula one look in local coordinates $z \xrightarrow{f} z^d$ to the form $\omega = w^n dw$,

$$f^\omega = (z^d)^n d(z^d) = dz^{dn+d-1}dz,$$

hence $\operatorname{ord}_x(f^*\omega) = dn + d - 1 = l(x)\operatorname{ord}_{f(x)}\omega + (l(x)-1)$.

$$\begin{aligned}
2g(X)-2 &= \deg(K_X)\\
&= \deg(\operatorname{div}(f^*\omega))\\
&= \sum_{x\in X}\operatorname{ord}_x(f^*\omega)\\
&= \sum_{x\in X} l(x)\cdot\operatorname{ord}_{f(x)}\omega + \sum_{x\in X}(l(x)-1)\\
&= \sum_{y\in Y}\Big(\sum_{x\in f^{-1}(y)} l(x)\Big)\operatorname{ord}_y\omega + \sum_{x\in X}(l(x)-1)\\
&= \deg(f)\sum_{y\in Y}\operatorname{ord}_y\omega + \sum_{x\in X}(l(x)-1)\\
&= \deg(f)(2g(Y)-2) + \sum_{x\in X}(l(x)-1).
\end{aligned}$$

□

Remark 3.5. *How to find a triangulation of a compact Riemann surface Y?*

One can get a triangulation of Y by using $f : Y : \mathbb{P}^1(\mathbb{C})$ a holomorphic map ($f \in \mathcal{M}(Y)\setminus\mathcal{O}(Y)$ do exist) and then pull-back a triangulation of $\mathbb{S}^2$ with vertices at $f(\mathcal{R})$ where $\mathcal{R}$ is the ramification points.

Topological proof of Riemann-Hurwitz theorem. Now take a triangulation of Y and add vertices at points in $f(\mathcal{R})$, then pull back through f this triangulation on a triangulation on X, then

$$\begin{aligned}
2-2g(X) &= \#\text{ vertices in } X - \#\text{ edges in } X + \#\text{ faces in } X\\
&= \big(\deg(f)\cdot(\#\text{ vertices in } Y) - \sum_{x\in X}(l(x)-1)\big)\\
&\qquad - (\deg(f))(\#\text{ edges in } X) + (\deg(f))(\#\text{ faces in } X)\\
&= \deg(f)(2-2g(Y)) - \sum_{x\in X}(l(x)-1).
\end{aligned}$$

□

Theorem 3.7 (Topological invariance of "g"). *Let X be a compact Riemann surface such that* $\dim H^1(X,\mathcal{O}) = \dim H^0(X,\Omega) = g$, *Then* $\dim H^1(X,\mathbb{C}) = 2g$.

Proof. We will construct a short exact sequence:

$$0 \to \Omega(X) \xrightarrow{\alpha} Z^1(X)/d\mathcal{E}(X) = H^1(X,\mathbb{C}) \xrightarrow{\tilde{\beta}} \mathcal{E}^{0,1}(X)/\overline{\partial}\mathcal{E}(X) = H^1(X,\mathcal{O}) \to 0.$$

Then from $\dim \Omega(X) = \dim H^1(X,\mathcal{O}) = g$, we have $\dim H^1(X,\mathbb{C}) = 2g$.

$$\Omega(X) \hookrightarrow Z^1(X) \to Z^1(X)/d\mathcal{E}(X),$$

then α is the composition of the inclusion with the quotient map.

$$\beta : Z^1(X) \to \mathcal{E}^{0,1}(X), \quad \eta \mapsto \eta^{(0,1)}.$$

β descends on $\tilde{\beta} : Z^1(X)/d\mathcal{E}(X) \to \mathcal{E}^{0,1}(X)/\overline{\partial}\mathcal{E}(X)$, because $\beta \circ d = \overline{\partial}$.

Let us prove that this gives an exact sequence.

α injective: let $\omega \in \Omega(X)$, $\alpha(\omega) = 0$. Then there exists $f \in \mathcal{E}(X)$ s.t. $\omega = df$. But ω is of type $(1,0)$, i.e. $\overline{\partial} f = 0$ hence $f \in \mathcal{O}(X)$ hence constant, then $\omega = df = 0$.

$\operatorname{Im} \alpha \subset \operatorname{Ker} \tilde{\beta}$: For any $\omega \in \Omega(X)$, $\alpha(\omega)$ is a holomorphic 1 form, hence its $(0,1)$ part is 0, which implies $\alpha(\omega) \in \operatorname{Ker} \tilde{\beta}$.

$\operatorname{Ker} \tilde{\beta} \subset \operatorname{Im} \alpha$: Take $u \in Z^1(X)$ such that $u + d\mathcal{E}(X)$ is in the kernel of $\tilde{\beta}$, i.e. $\tilde{\beta}(u + d\mathcal{E}(X)) = 0$, i.e. $\beta u = \overline{\partial} v$ for some $v \in \mathcal{E}(X)$. Then $u - dv$ is holomorphic because it is closed and of type $(1,0)$, hence $u + d(\mathcal{E}(X)) \in \alpha(\Omega)$.

For $\tilde{\beta}$ is surjective we need

Lemma 3.1 (**Weyl lemma**). *For any $u \in \mathcal{E}^{0,1}(X)$, there exits $v \in \mathcal{E}(X)$ such that $u - \overline{\partial} v$ is closed.*

Proof of the lemma. Use the fact that $\partial\overline{\partial} : \mathcal{E}(X) \to \mathcal{E}^2(X)$ has an image which is given by $\sigma \in \mathcal{E}^2(X)$ such that $\int_X \sigma = 0$.

With this we have that $du = \partial\overline{\partial} v$, $v \in \mathcal{E}(X)$ (since $\int_X du = 0$ by stokes). This implies $d(u - \overline{\partial} v) = du - \partial\overline{\partial} v = 0$. □

This proves the surjectivity of $\tilde{\beta}$. For any $u \in \mathcal{E}^{0,1}(X)$ we need to show that there exists $\overline{u} \in Z^1(X)$ and a function $v \in \mathcal{E}(X)$ such that $u = \beta(\overline{u}) + \overline{\partial} v$.

Weyl lemma insures that $\overline{u}$ can be chosen of type $(0,1)$, closed and anti-holomorphic. Therefor $\tilde{\beta}|_{\overline{\Omega}(X)}$ is surjective. □

Proposition 3.9. *Let X be a compact Riemann surface and the maps defined as*

$$\begin{array}{ccc} \Omega(X) & \xrightarrow{\alpha} & H^1(X,\mathbb{C}) \\ \downarrow & & \downarrow \cong \\ Z^1(X) & \longrightarrow & Z^1(X)/d\mathcal{E}(X). \end{array} \qquad \begin{array}{ccc} \overline{\Omega}(X) & \xrightarrow{\overline{\alpha}} & H^1(X,\mathbb{C}) \\ \downarrow & & \downarrow \cong \\ Z^1(X) & \longrightarrow & Z^1(X)/d\mathcal{E}(X). \end{array}$$

We have that α and $\tilde{\alpha}$ are injective and

$$H^1(X,\mathbb{C}) = \alpha(\Omega(X)) \oplus \overline{\alpha}(\overline{\Omega}(X)).$$

Proof. We already proved that α is injective (same proof implies $\overline{\alpha}$ is injective). $\dim \alpha(\Omega(X)) = \dim \overline{\alpha}(\overline{\Omega}(X)) = g$and we proved that $\dim H^1(X, \mathbb{C}) = 2g$.

We need to prove that $\alpha(\Omega(X)) \cap \overline{\alpha}(\overline{\Omega}(X)) = \{0\}$. If $\omega \in \Omega(X)$ and $\omega' \in \overline{\Omega}(X)$ such that $\alpha(\omega) = \overline{\alpha}(\omega')$ then there is $f \in \mathcal{E}(X)$ s.t. $\omega - \omega' = df$. Then $\omega = \partial f$ and $\omega' = -\overline{\partial} f$. f is harmonic since $\overline{\partial}\partial f = \overline{\partial}\omega = 0$ since ω is holomorphic. Then f constant and $\omega = \omega'$ hence $= 0$. ☐

Remark 3.6. *Cohomology type.*

3.5 Abel theorem

Theorem 3.8. *For any compact Riemann surface of genus $g \geq 1$ and for any $P \in X$, there is $\omega \in H^0(X, \Omega)$ such that $\omega(P) \neq 0$.*

Proof. Assume that all $\omega \in \Omega(X)$ such that $\omega(P) = 0$ for some $P \in X$. Then $H^0(X, \Omega_{-P}) = H^0(X, \mathcal{O}_K \otimes \mathcal{O}(-P)) \hookrightarrow H^0(X, \Omega)$ is a group isomorphism.

Applying Riemann-Roch theorem for both $\mathcal{O}_K$ and $\mathcal{O}_K \otimes \mathcal{O}_{-P}$ implies that

$$\dim H^1(X, \mathcal{O}_K \otimes \mathcal{O}_{-P}) = \dim H^1(X, \mathcal{O}_K) + 1.$$

By Serre duality

$$\dim H^1(X, \mathcal{O}_K) = \dim H^0(X, \mathcal{O}) = 1.$$

Then $\dim H^1(X, \mathcal{O}_K \otimes \mathcal{O}_{-P}) = 2$, hence again by Serre duality,

$$\dim H^0(X, \mathcal{O}_P) = 2.$$

Thus there is a meromorphic function $f \in \mathcal{M}(X)$ having a pole of order at most 1 at point P.

This implies there is a well-defined map $X \to \mathbb{P}^1$, a contradiction to $g = 1$. ☐

Remark 3.7. *This implies there is a well-defined map*

$$X \to \mathbb{P}^{g-1}, \quad x \mapsto [\omega_1(x) : \cdots : \omega_g(x)],$$

where $(\omega_1, \cdots, \omega_g)$ is a basis of $\Omega(X)$.

You can associate to any $x \in X$ the hyperplane in $\Omega(X)$ of those $\omega \in \Omega(X)$ such that $\omega(x) = 0$.

For X compact Riemann surface and a divisor $\sum_{i=1}^{k} n_i \cdot P_i$, for $n_i \in \mathbb{Z}$ and $P_i \in X$, which is now called 0-chains. Then we introduce the 1-chains $c = \sum_{i=1}^{k} n_i c_i$ with $n_i \in \mathbb{Z}$ and $c_i : [0,1] \to X$. One can define for any $\omega \in \mathcal{E}^1(X)$,

$$\int_c \omega = \sum_{i=1}^{k} n_i \cdot \int_{c_i} \omega.$$

The set of 1-chains $C_1(X)$ is an abelian group. There is a border map:

$$\partial : C_1(X) \to C_0(X) = \mathrm{Div}(X), c = \sum_{i=1}^{k} n_i c_i \mapsto c = \sum_{i=1}^{k} n_i(c_i(1) - c_i(0)).$$

If c is a closed curve, then $\partial c = 0$.

If $c \in C_1(X)$, then $\deg(\partial c) = 0$. Then the image of $C_1(X)$ is in $\mathrm{Div}_0(X)$, the group of divisors of degree 0 on X. Moreover, $\partial(C_1(X)) = \mathrm{Div}_0(X)$, since for any $D \in \mathrm{Div}_0(X)$, one can consider pairs of points (P_i, Q_i) such that $D = \sum P_i - \sum Q_i$, then let c_i be a curve with $c_i(1) = P_i$ and $c_i(0) = Q_i$.

We define 1-cycles

$$Z_1(X) = \mathrm{Ker}(C_1(X) \xrightarrow{\partial} \mathrm{Div}_0(X)).$$

We say $c, c' \in Z_1(X)$ are homologous if $\forall \omega \in Z^1(X)$ a closed smooth 1-form, $\int_c \omega = \int_{c'} \omega$. Now we define

$$H_1(X, \mathbb{Z}) = Z_1(X)/\sim .$$

Remark 3.8. *For any $\gamma \in H_1(X, \mathbb{Z})$ and any $\omega \in Z^1(X)$, $\int_\gamma \omega$ is well-defined.*

Remark 3.9. *For any closed homotopic curves c_1 and c_2, $\int_{c_1} \omega_1 = \int_{c_2} \omega$, $\forall \omega \in Z^1(X)$. Since we have*

$$0 = \int_{c\times[0,1]} d\omega = \int_{c_1} \omega - \int_{c_2} \omega.$$

Thus i is well-defined in the following diagram and i is always surjective,

$$\begin{tikzcd} \pi_1(X) \arrow[r, "i"] \arrow[d] & H_1(X,\mathbb{Z}) \\ \pi_1(X)/[\pi_1(X),\pi_1(X)] \arrow[ur] & \end{tikzcd} .$$

Usually i is not injective since $\pi_1(X)$ is not abelian in general. Moreover, in this case, we have $\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X, \mathbb{Z})$ (but we will not prove it).

Theorem 3.9 (Jacobian of a Riemann surface)**.** *Integration over* 1*-cycles determinus*

$$\int : H_1(X, \mathbb{Z}) \to (\Omega(X))^*,$$

such that the image is a lattice called the period lattice.

$$(\Omega(X))^*/H_1(X, \mathbb{Z}) = (H^0(X, \Omega))^*/H_1(X, Z) \cong \mathbb{C}^g/\Lambda =: \mathrm{Jac}(X),$$

is called the jacobian of X (topologically $\mathrm{Jac}(X) \cong (\mathbb{S}^1)^{2g}$).

Example 3.14. *$X = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ an elliptic curve. $\Omega(X) = \mathbb{C}dz$, the period lattice is $\mathbb{Z} \oplus \tau\mathbb{Z}$, and $\mathrm{Jac}(X) = X$.*

Remark 3.10. *Integration map associated to any $[\gamma] \in H_1(X, \mathbb{Z})$ the* 1*-form on $\Omega(X)$ is the integration over γ, $\int_\gamma \in (\Omega(X))^*$ such that $\omega \mapsto \int_\gamma \omega$ for any $\omega \in \Omega(X)$.*

Remark 3.11. *Assume* $\int_\gamma \omega = 0$ *for any* $\gamma \in H_1(X,\mathbb{Z})$*, this implies that* ω *admits a primitive which is a holomorphic function, hence constant hence* $\omega = 0$*.*

Remark 3.12. $H_1(X,\mathbb{Z}) \hookrightarrow H_1(X,\mathbb{C})$ *(is not injective in general).* $H^1_{DR}(X,\mathbb{C}) = \Omega(X) \oplus \overline{\Omega(X)}$*.*

The image of $H_1(X,\mathbb{Z})$ *in* $H_1(X,\mathbb{C})$ *is discrete hence it's discrete in* $(\Omega(X))^*$*.*

Assume that $\int$ *sends* $H_1(X,\mathbb{Z})$ *in a real hyperplane in* $(\Omega(X))^*$*. This implies there is* $\omega \in \Omega(X)$*, such that* $\mathrm{Re}(\int_\gamma \omega) = 0$ *for any* $\gamma \in H_1(X,\mathbb{Z})$*. Then* $\mathrm{Re}(\int_\gamma \omega)$ *is a well defined function on* X*, then* $\int_\gamma \mathrm{Re}\,\omega$ *is a well defined function on* X*.*

Consider the universal cover of X *and since* $\int_\gamma \mathrm{Re}\,\omega = 0$*, there is a real harmonic primitive* f *on* $\tilde{X}$ *of* $\mathrm{Re}\,\omega$ *such that* $\int_\gamma \mathrm{Re}\,\omega = f$*. Moreover,* f *is well-defined on* X*, hence by maximal principle,* f *is constant, hence* $\mathrm{Re}\,\omega = 0$*.*

$\mathrm{Im}\,\omega = J\,\mathrm{Re}\,\omega$ *hence* $\omega = 0$*.*

Another interpretation is take $(\omega_1, \cdots, \omega_g)$ a torus of $\Omega(X)$, then

$$H_1(X,\mathbb{Z}) \to \mathbb{C}^g, \quad \gamma \mapsto (\int_\gamma \omega_1, \cdots, \int_\gamma \omega_g).$$

The image of this map is the period lattice $\mathrm{Jac}(X) = \mathbb{C}^g/\text{Period Lattice}$.

There is a canonical embedding $X \to \mathrm{Jac}(X)$. Fix a basis point $O \in X$, consider the map

$$P \in X \to (\omega \to \int_{\gamma_{OP}} \omega) \in (\Omega(X))^*.$$

This is not well-defined, since if we choose two curves connecting O and P, we have

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\gamma \cup (-\gamma')} \omega \in H_1(X,\mathbb{Z}),$$

hence $\int_{\gamma_{OP}} \in (\Omega(X))^*/\text{Period Lattice} = \mathrm{Jac}(X)$.

Theorem 3.10 (Abel-Jacobi theorem). *The space of holomorphic line bundles of degree* 0 *over* X *is canonically isomorphic to* $\mathrm{Jac}(X)$ *through the map*

$$\mathrm{Pic}_0(X) := \mathrm{Div}_0(X)/\sim \to \mathrm{Jac}(X), \quad D = \sum_{i=1}^k (P_i - Q_i) \mapsto (\omega \to \int_c \omega) \in (\Omega(X))^*,$$

where c *is a* 1*-chain such that* $\partial c = D$*.*

Remark 3.13. *The map is well-defined because if*

$$\int_{c_1} \omega - \int_{c_2} \omega = \int_{c_1 \cup (-c_2)} \omega \in \textit{Periods}.$$

If $D = D' + (f)$*, the map is well defined because we will see that Abel-Jacobi map send* (f) *to* 0*.*

Proof of Abel-Jacobi theorem.

Step 1. For $f \in \mathcal{M}(X)$, $f : X \to \mathbb{P}^1$, we will prove $\Phi(\mathrm{div}(f)) = 0$. Set $R \subset X$ the ramified set of f: $R = \{x \in X : f'(x) = 0\}$, i.e. $l(x) > 0$. f defines a cover from $X \setminus f^{-1}(f(R)) \to \mathbb{P}^1 \setminus f(R)$. For any $y \in Y = \mathbb{P}^1 \setminus R$, there are open sets $V \ni y$ and $U_i \subset X \setminus f^{-1}(f(R))$, such that $f^{-1}(V) = \bigcup_{i=1}^{n} U_i$ and $forall i$, $f|_{U_i} : U_i \to V$ is a biholomorphism. Let us denote by $\varphi_i = (f|_{U_i})^{-1}$ and $\forall \omega \in \Omega(X)$ define $\omega_i = \varphi_i^* \omega|_{U_i} \in \Omega(V)$, define $\mathrm{tr}(\omega) = \sum_{i=1}^{n} \omega_i \in \Omega(V)$.

We define $\mathrm{tr}(\omega) \in \Omega(\mathbb{P}^1 \setminus f(R))$. Since $\omega \in \Omega(X)$, ω is bounded in the neighborhood of points in $f^{-1}(f(R))$, $\omega = h(z)dz$ with h bounded. This implies that in the neighborhood of points $P \in f(R)$, $\mathrm{tr}(\omega)$ is bounded: $l(z)dz$, l bounded in the neighborhood of P in $\mathbb{P}^1 \setminus f(R)$. By Riemann removing singularity theorem, $\mathrm{tr}(\omega)$ extends holomorphically to a section of $\Omega(\mathbb{P}^1) = \{0\}$ hence $\mathrm{tr}(\omega) = 0$.

Let γ be a curve in $\mathbb{P}^1$ such that $\gamma(0) = 0$ and $\gamma(1) = \infty$. Denote $f^{-1}(\gamma) = c_1 + \cdots + c_n =: c$ such that $\partial(c) = (f)$. Then

$$\int_c \omega = \int_\gamma \mathrm{tr}(\omega) = 0,$$

hence $\Phi(\mathrm{div}(f)) = 0$.

Step 2, Abel theorem. $\Phi(0) = 0$ implies $D = \mathrm{div}(f)$, for some $f \in \mathcal{M}(X)$.

Step 3, Jacobi inversion theorem. Φ is surjective.

Idea of the proof. Fix $(Q_1, \cdots, Q_g) \in X \times \cdots \times X$ g-fold. Define a map

$$F : X \times \cdots \times X \to \mathrm{Jac}(X), \quad (P_1, \cdots, P_g) \mapsto (\omega \to \int_c \omega),$$

where c is a 1-chain in X such that $\partial c = P_1 + \cdots + P_g - (Q_1 + \cdots + Q_g)$. By the same reason, it does not depend on the choice of c.

$$dF : TX \times \cdots \times TX \to (\Omega(X))^* \Rightarrow (dF)^* : \Omega(X) \to T^*X \times \cdots \times T^*X,$$

and $(dF)^*(\omega) = (\omega(P_1), \cdots, \omega(P_g))$.

Exercise 3.6. *Fix points $P_1, \cdots, P_g$, $\omega(P_1) = \cdots = \omega(P_g) = 0$, we have $\omega \equiv 0$.*

Then $(dF)^*$ is an injection.

Recall that for X, Y compact connected complex manifolds of same dimension, $f : X \to Y$ holomorphic is surjective iff there exists $x \in X$ such that $df(x)$ is invertible.

Because in this case, by local inverse theorem there exists (U, x) and $(V, f(x))$ such that $f|_U : U \to V$ is a biholomorphism. This implies that V contains at least one regular value y and for this value $\#f^{-1}(y) \geq 1$, hence $\deg f \geq 1$, hence surjective and for regular values, $\deg f = \#f^{-1}(y)$. □

Example 3.15. *There is a particular case of this when X is of genus 1. Hence $\dim \Omega(X) = 1$ and take $\omega \in \Omega(X) \setminus \{0\}$. Fix $P \in X$, consider $X \to \mathrm{Jac}(X) = \mathbb{C}/\Lambda$, $Q \mapsto \int_c \omega$, where $c(0) = P$ and $c(1) = Q$.*

$\Phi : X \to \mathbb{C}/\Lambda$, and $d\Phi = \omega$. Assume by contradiction Φ is not injective, then by Abel theorem, there is $f \in \mathcal{M}(X)$, $(f) = (P) - (Q)$, hence f admits a unique simple pole, then $f : X \cong \mathbb{P}^1$, a contradiction since $g = 1$.

In the devoir, ω does not vanish, hence this map Φ such that $d\Phi = \omega$ does not vanish, hence Φ is locally injective and hence a cover. A cover of an elliptic curve is an elliptic curve.

How to prove that Λ is a lattice in $\mathbb{C}$.

$|\omega|^2$ is a Riemannian metric locally isomorphic to $|dz|^2 = dx^2 + dy^2$. There is a map $(\tilde{X}, |\tilde{\omega}|^2) \xrightarrow{\Phi} (\mathbb{C}, |dz|^2)$ which is an isometry, since $|\omega|^2$ is geodesically complete on X, $|\tilde{\omega}|^2$ is geodesically complete on $\tilde{X}$ hence Φ is an isomorphism.

Dual point of viewL there is $X \in H^0(TX)$ such that $\omega(X) \equiv 1$. The flow of X is complete: there is $f : \mathbb{C} \times X \to X$ with open orbits, hence there is only one orbit $X = \mathbb{C}/\operatorname{Stab}(P)$. $\operatorname{Stab}(P)$ is a discrete subgroup such that the quotient is compact, hence $\operatorname{Stab}(P)$ is a lattice.

Theorem 3.11 (**Tischeler theorem**)**.** *Let M be a compact real manifold and $\omega \in Z^1(M)$ a closed one-form, which does not vanish on M. Then there is a submersion $M \to \mathbb{S}^1$.*

Proof. $H_1(M, \mathbb{Z}) \to (Z^1(M))^*$, $[\gamma] \mapsto [\omega \to \int_\gamma \omega]$, consider the periods of ω.

The image of $\int_\omega$ is the obstruction for ω to admit a primitive, in $H^1(M, \mathbb{R})$. By a slight deformation of $\omega \in H^1_{DR}(M, \mathbb{R})$ we can assume that ω_1 is close to ω, does not vanish and admits rational periods (because $H^1(M, \mathbb{Q})$ dense in $H^1(M, \mathbb{R})$). Multiplying by some $n \in \mathbb{Z}$, $n\omega_1$ will have integer periods.

$$\begin{tikzcd} \tilde{M} \arrow[r, "\int \omega = f"] \arrow[d] & \mathbb{R} \arrow[d] \\ M \arrow[r, "\text{fibration}"] & \mathbb{R}/\mathbb{Z} \end{tikzcd},$$

since $df = \omega \neq 0$.

The action of $\pi_1(M)$ on $\tilde{M}$ gives

$$f(\gamma \cdot m) = \rho([\gamma]) + f(m),$$

where $\gamma \in \pi_1(M)$ and $\rho([\gamma]) = \int_\gamma \omega$. □

Theorem 3.12 (Suale)**.** *$Diff^+(\mathbb{S}^2)$ has the same homotopy type as $SO(3, \mathbb{R})$.*

Proof. Any smooth vector field on $\mathbb{S}^2$ gives a 1-parameter family of diffeomorphisms.

Idea of proof (earle-Eells): Construct the sapce of complex structures on $\mathbb{S}^2$ compatible with the orientation.

$$\mathrm{Comp}^+ = \{j \in H^0(\mathrm{End}(T\mathbb{S}^2)) : j \circ j = -\mathrm{id},\ \text{orientation}\}.$$

By isothermal coordinates theorem they are all integrable and define complex structure.

$Diff^+(\mathbb{S}^2)$ acts naturally on Comp^+, $\forall \varphi \in Diff^+(\mathbb{S}^2)$ and $j \in \mathrm{Comp}^+$, define $\varphi^* j \in \mathrm{Comp}^+$.

Uniformization theorem implies there is only one orbit: $\forall j \in \mathrm{Comp}^+$, there is $\varphi \in Diff^+(\mathbb{S}^2)$ such that $j = \varphi^* j_0$, j_0 is the standard complex structure. Moreover,

$$\mathrm{Stab}(j_0) = \{\varphi \in Diff^+ : \varphi^* j_0 = j_0\} = \mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) \cong \mathrm{PSL}(2, \mathbb{C}).$$

Thus

$$\mathrm{Comp}^+ \cong Diff^+(\mathbb{S}^2)/\,\mathrm{PGL}(2, \mathbb{C}).$$

Comp^+ is the space of sections $\{s \in H^0(\mathrm{End}(T\mathbb{S}^2)) : j^2 = -\mathrm{id}\}$ of a bundle over $\mathbb{S}^2$ whose fiber is the hyperbolic plane.

Topology: The space of sections of a bundle of contractible fiber is contractible

$$Diff^+(\mathbb{S}^2) \cong PGL(2, \mathbb{C}) \times \mathrm{Comp}^+(\mathbb{S}^2),$$

hence $Diff^+(\mathbb{S}^2)$ has the same homotopy type as $PGL(2, \mathbb{C})$ and $PGL(2, \mathbb{C})$ is homeomorphic to $\mathrm{SO}(3, \mathbb{R}) \times \mathbb{R}^2$, hence has the same homotopy type as $\mathrm{SO}(3, \mathbb{R})$.

$\square$